

Strategic Experimentation with Asymmetric Players*

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Abstract

We examine a two-player game with two-armed exponential bandits a la [Keller, Rady, Cripps (2005)], where players operate different technologies for exploring the risky option. We characterise the set of Markov perfect equilibria, and show that there always exists an equilibrium in which the player with the inferior technology uses a cutoff strategy. All Markov perfect equilibria imply the same *amount* of experimentation but differ with respect to the expected speed of the resolution of uncertainty. If and only if the degree of asymmetry between the players is high enough, there exists a Markov perfect equilibrium in which both players use cutoff strategies. Whenever this equilibrium exists, it welfare dominates all other equilibria. This contrasts with the case of symmetric players, where there never exists a Markov perfect equilibrium in cutoff strategies. We also show that the equilibrium where only the player with the inferior technology uses a cutoff strategy is not welfare dominated.

JEL Classification Numbers: C73, D83, O31 **Keywords:** Two-armed Bandit, Heterogeneous Agents, Free-Riding, Learning

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1 Introduction

In many instances, the information produced by one agent is interesting to other agents as well. Think e.g. of firms exploring neighbouring oil patches: If one firm strikes oil, chances are there will be oil in its neighbour's patch as well. Such games of purely informational externalities have been analysed by the strategic bandit literature,¹ which so far has only analysed the case of homogeneous agents. However, in many instances, one of the oil firms, for example, might be a big multinational firm that has access to a superior drilling technology. In this article, we aim to analyse the impact of asymmetries in players' exploration technologies in a game of strategic experimentation with two-armed exponential bandits.

The seminal paper by [Keller, Rady, Cripps (2005)] analyses this problem with homogeneous players. In the current paper, we generalise the analysis by introducing asymmetric players, in the sense that their payoff arrival rates from a good risky arm differ. This implies that, given the risky arm is good, the expected time needed to learn this differs between the players. As actions and outcomes are perfectly publicly observable, and players start out with a common prior, they will always have a common posterior belief. We characterise the set of Markov perfect equilibria with the players' common posterior belief as the state variable for all ranges of asymmetry between the players. If the degree of asymmetry between the players is sufficiently high, there exists an *equilibrium in cutoff strategies*, i.e. where both players use a cutoff strategy. That is, either player uses the risky arm if and only if the likelihood he attributes to the option being good is greater than a certain threshold. This equilibrium is unique in the class of equilibria in cutoff strategies. Whenever only one of the players experiments and the other free rides in this equilibrium, it is always the player with the weaker technology who free rides. In the case of homogeneous players ([Keller, Rady, Cripps (2005)]), by contrast, there never exists an equilibrium in cutoff strategies, and players swap the roles of pioneer and free-rider at least once in any equilibrium. In our setting, aggregate payoffs in the equilibrium in cutoff strategies are higher than in any other equilibrium. If the degree of asymmetry is low, at least one player uses a non-cutoff strategy in any equilibrium. In contrast to the homogeneous case ([Keller, Rady, Cripps (2005)]), we furthermore show that more frequent switches of arms do not unambiguously improve the equilibrium welfare with asymmetric players.

This paper contributes to the literature on strategic experimentation with bandits, a problem studied quite widely in economics, amongst others, by [Keller and Rady (2010)], [Klein and Rady (2011)]

¹The first paper to do so was [Bolton and Harris (1999)]. [Keller, Rady, Cripps (2005)] have introduced exponential bandits, which we shall use here.

and [Thomas (2017)]. In all of these papers, players are homogeneous. Except in [Thomas (2017)] and [Klein and Rady (2011)], players' bandits are of the same type and *free-riding* is a common feature in all the above models except for [Thomas (2017)]. Many variants of this problem have been studied in the literature. [Rosenberg, Salomon, Vieille (2013)] and [Murto and Välimäki (2011)], for instance, assume that switches to the safe arm are irreversible and that experimentation outcomes are private information, while [Bonatti and Hörner (2011)] and [Heidhues, Rady, Strack (2015)] investigate the case of private actions. In [Dong (2018)], actions and outcomes are public, but one of the players receives an initial private signal. [Rosenberg, Solan, Vieille (2007)] analyse the role of the observability of outcomes and the correlation between risky-arm types in a setting in which a switch to the safe arm is irreversible. [Besanko and Wu (2013)] use the [Keller, Rady, Cripps (2005)] framework to study how an R&D race is impacted by market structure. The paper closest to the present paper is [Keller, Rady, Cripps (2005)], who find that, with homogeneous players, there is never an equilibrium in cutoff strategies. By contrast, we show that, with heterogeneous players, an equilibrium in cutoff strategies may exist, and that it is welfare-maximising whenever it exists.

The rest of the paper is organised as follows. Section 2 sets out the model. Section 3 discusses the social planner's solution. A detailed analysis of equilibria for different ranges of heterogeneity is undertaken in Section 4. Finally, Section 5 concludes. Payoff functions are shown in Appendix A, while some proofs are relegated to Appendix B.

2 Two armed bandit model with heterogeneous players

There are two players (1 and 2), each of whom faces a two-armed bandit in continuous time. One of the arms is safe, in that a player who uses it gets a flow payoff of $s > 0$. The risky arm can be either good or bad. Both players' risky arms are of the same type. If the risky arm is good, then a player using it receives a lump sum, drawn from a time-invariant distribution with mean $h > s$, at the jumping times of a Poisson process. The Poisson process governing player 1's arrivals has intensity $\lambda_1 = 1$, while player 2's arrive according to a Poisson process with intensity $\lambda_2 \in (\frac{s}{h}, 1)$. Thus, a good risky arm gives player 1 (2) an expected payoff flow of $g_1 = \lambda_1 h = h$ ($g_2 = \lambda_2 h$), with $g_1 > g_2 > s$. The parameters and the game are common knowledge.

The uncertainty in this model arises from the fact that players do not initially know whether their risky arms are good or bad. Players start with a common prior belief $p_0 \in (0, 1)$ that their risky arms are good. Players have to decide in continuous time whether to choose the safe arm or the risky arm. At each instant, players can choose only one arm. We write $k_{i,t} = 1$ ($k_{i,t} = 0$) if player $i \in \{1, 2\}$ uses his risky (safe) arm at instant $t \geq 0$. Players' actions and outcomes are publicly observable and, based

on these, they update their beliefs. Players discount the future according to the common discount rate $r > 0$.

Let p_t be the players' common belief that their risky arms are good at time $t \geq 0$. Given player i 's ($i \in \{1, 2\}$) actions $\{k_{i,t}\}_{t \geq 0}$, which are required to be progressively measurable with respect to the available information and to satisfy $k_i(t) \in \{0, 1\}$ for all $t \geq 0$, player i 's expected payoff is given by

$$\mathbb{E} \left[\int_0^\infty r e^{-rt} [(1 - k_{i,t})s + k_{i,t} p_t g_i] dt \right],$$

where the expectation is taken with respect to the processes $\{k_{i,t}\}_{t \geq 0}$ and $\{p_t\}_{t \geq 0}$. As can be seen from the objective function, there are no payoff externalities between the players. Indeed, the presence of the other player impacts a given player's payoffs only via the information that he generates, i.e. via the belief.

As mentioned in the Introduction, we will focus our analysis on Markov perfect equilibria with the players' common posterior belief as the state variable. Formally, a Markov strategy of player i is any left-continuous function $k_i : [0, 1] \rightarrow \{0, 1\}$, $p \mapsto k_i(p)$ ($i = 1, 2$) that is also piecewise continuous, i.e. continuous at all but a finite number of points.

As only a good risky arm can yield positive payoffs in the form of lump sums, the arrival of a lump sum fully reveals the risky arm to be good. Hence, if either player receives a lump sum at a time $\tau \geq 0$, then $p_t = 1$ for all $t > \tau$. In the absence of a lump-sum arrival, the belief follows the following law of motion for a.a. t :

$$dp_t = -(k_{1,t} + \lambda_2 k_{2,t}) p_t (1 - p_t) dt.$$

3 Planner's Problem

Suppose there is a benevolent social planner, who controls the actions of both players and wants to maximise the sum of their payoffs. Since the planner's expected payoff at any point in time only depends on the belief at that time and the belief follows a controlled Markov process, this is a Markov decision problem. Therefore, it is without loss of generality for the planner to restrict himself to Markov strategies $(k_1(p_t), k_2(p_t))$ with the posterior belief p_t as the state variable. The Bellman equation for the planner's problem is given by

$$v(p) = 2s + \max_{k_1, k_2 \in \{0, 1\}} \{k_1 [B_1(p, v) - c_1(p)] + k_2 [B_2(p, v) - c_2(p)]\}, \quad (1)$$

where we write $v(p)$ for the planner's value function, and, like [Keller, Rady, Cripps (2005)], define the myopic opportunity cost of having player i play risky, $c_i(p) = s - pg_i$, and the corresponding learning benefit

$$B_i(p, v) = p \frac{\lambda_i}{r} \{(g_1 + g_2) - v(p) - v'(p)(1-p)\}.$$

Note that the planner's Bellman equation is linear in both k_1 and k_2 , so that our restriction to action plans $\{(k_{1,t}, k_{2,t})\}_{t \geq 0}$ with $k_{i,t} \in \{0, 1\}$ for all (i, t) is without loss in the planner's problem. To state the following proposition, which describes the planner's solution, we define $g = g_1 + g_2$, $\lambda = 1 + \lambda_2$, $\mu = \frac{r}{\lambda}$, $u_1(p) := (1-p) \left(\frac{1-p}{p}\right)^r$, $u_0(p) := (1-p) \left(\frac{1-p}{p}\right)^\mu$.

Proposition 1 *The planner's optimal policy $k^*(p) = (k_1^*, k_2^*)(p)$ is given by*

$$(k_1^*, k_2^*)(p) = \begin{cases} (1, 1) & \text{if } p \in (p_2^*, 1) \\ (1, 0) & \text{if } p \in (p_1^*, p_2^*] \\ (0, 0) & \text{if } p \in (0, p_1^*] \end{cases}$$

and the value function is

$$v(p) = \begin{cases} gp + \left[\frac{\lambda}{\lambda_2} s - gp_2^* \right] \frac{u_0(p)}{u_0(p_2^*)} & \text{if } p \in (p_2^*, 1], \\ s + \left[\frac{g+rg_1}{1+r} - \frac{s}{1+r} \right] p + \left[s - \left(\frac{g+rg_1}{1+r} - \frac{s}{1+r} \right) p_1^* \right] \frac{u_1(p)}{u_1(p_1^*)} & \text{if } p \in (p_1^*, p_2^*], \\ 2s & \text{if } p \in (0, p_1^*], \end{cases}$$

where p_1^* is defined as

$$p_1^* = \frac{rs}{(1+r)g_1 + g_2 - 2s}, \quad (2)$$

and $p_2^* \in (p_1^*, \frac{s}{g_2})$ is implicitly defined by $v(p_2^*) = \frac{\lambda}{\lambda_2} s$.

Proof. Proof is by a standard verification argument. Please see the Appendix B.1 for details. ■

By the above proposition, the belief at which player 1 switches to the safe arm in the planner's solution is higher than it would be if both players' Poisson arrival rates were equal to $\lambda_1 = 1$. This is because, as player 2's arrival rate λ_2 decreases, the benefit from player 1's experimentation decreases.

The planner's solution is depicted in the Figure 1. The planner's value function is a smooth convex curve which lies in the range $[2s, g]$. At the belief $p_2^*(p_1^*)$, player 2 (1) switches to the safe arm.

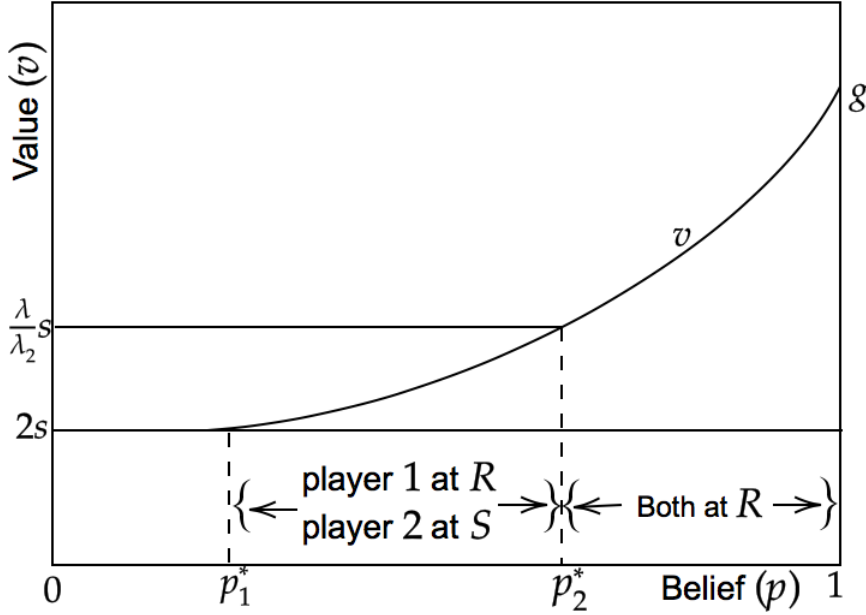


Figure 1

4 Non-cooperative game

We will first analyse a player's best responses to a given Markov strategy of the other player.

Best Responses: Fix player j 's strategy k_j ($j \in \{1, 2\} \setminus \{i\}$). If the payoff function from player i 's response satisfies the following Bellman equation, player i is playing a best response:²

$$v_i(p) = s + k_j(p)\lambda_j b_i(p, v_i) + \max_{k_i \in \{0, 1\}} k_i[\lambda_i b_i(p, v_i) - (s - g_i p)] \quad (3)$$

where

$$b_i(p, v_i) = p \frac{\{g_i - v_i - (1 - p)v_i'\}}{r}.$$

As before, $\lambda_i b_i(p, v_i)$ can be interpreted as the learning benefit accruing to player i due to his own experimentation, while $\lambda_j b_i(p, v_i)$ is the learning benefit accruing to player i from player j 's experimentation. The myopic opportunity cost of experimentation continues to be $c_i(p) = s - g_i p$.

For a given $k_j \in \{0, 1\}$, from (3) we know that player i 's payoff function satisfies the Bellman

²By standard results, on any open interval of beliefs in which player j 's action choice is constant, player i 's value function v_i will be continuously differentiable. At those (finitely many) beliefs at which player j 's action changes, v' should be understood as the left derivative of v (since beliefs can only drift down).

equation if and only if

$$k_i(p) \begin{cases} = 1 & \text{if } \lambda_i b_i(p, v_i) > s - g_i p, \\ \in \{0, 1\} & \text{if } \lambda_i b_i(p, v_i) = s - g_i p, \\ = 0 & \text{if } \lambda_i b_i(p, v_i) < s - g_i p. \end{cases}$$

If $\lambda_i b_i(p, v_i) > s - g_i p$ then $k_i = 1$ is the unique best response. From 3, we can conclude that this requires $v_i > s + k_j \lambda_j b_i(p, v_i) > s + k_j \frac{\lambda_j}{\lambda_i} (s - g_i p)$. A similar argument applies for the situations when the best responses are $k_i \in \{0, 1\}$ and $k_i = 0$ respectively. This allows us to infer that

$$k_i(p) \begin{cases} = 1 & \text{if } v_i > s + k_j \frac{\lambda_j}{\lambda_i} [s - g_i p], \\ \in \{0, 1\} & \text{if } v_i = s + k_j \frac{\lambda_j}{\lambda_i} [s - g_i p], \\ = 0 & \text{if } v_i < s + k_j \frac{\lambda_j}{\lambda_i} [s - g_i p]. \end{cases}$$

This implies that when $k_j = 1$, player i chooses the risky arm, safe arm or is indifferent between them depending on whether his value in the (p, v) plane lies above, below, or on the line

$$D_i(p) = s + \frac{\lambda_j}{\lambda_i} [s - g_i p] \quad (4)$$

The single-agent threshold for player i is given by

$$\bar{p}_i = \frac{\mu_i s}{\mu_i s + (1 + \mu_i)(g_i - s)} \quad (5)$$

where $\mu_i = \frac{r}{\lambda_i}$. In Appendix A.2, we display the ODEs the players' payoff functions satisfy, as well as their solutions, for each possible action profile. We start off by showing that, as in the homogeneous case ([Keller, Rady, Cripps (2005)]), no efficient equilibrium exists.

Proposition 2 *In any MPE, both players play safe at all beliefs in $[0, \bar{p}_1]$. There is thus no efficient MPE.*

Proof. Suppose to the contrary that p_l , the infimum of the set of beliefs at which at least one player plays risky, satisfies $p_l < \bar{p}_1$. Clearly, $v_i(p_l) = s$ for both $i \in \{1, 2\}$. We shall now distinguish two cases depending on whether or not there exists an $\bar{\varepsilon} > 0$ such that, in any ε -right neighbourhood of p_l with $\varepsilon \in (0, \bar{\varepsilon})$, only one player i plays risky. If there does not exist such an $\bar{\varepsilon} > 0$, i is not playing a best response, because $p_l < \bar{p}_i < \frac{s}{g_i}$ implies that the point (p_l, s) is below the diagonal D_i . In the other case, player i faces the same trade-off as a single agent, and does not play a best response either, because $p_l < \bar{p}_i$. ■

In the next subsection, we will characterise the condition under which an equilibrium in cutoff strategies exists.

4.1 Equilibrium in cutoff strategies

As we have argued in the proof of Proposition 2, there is no experimentation below the belief \bar{p}_1 in any equilibrium. We will now argue that, in any equilibrium, only player 1 will experiment in some right-neighbourhood of \bar{p}_1 , implying that player 1 is the last player to experiment in any equilibrium.

By Proposition 2, we know that $v_1(\bar{p}_1) = v_2(\bar{p}_1) = s$, and thus, by continuity, both players' value functions must be below their respective diagonals D_i in some neighbourhood of \bar{p}_1 . Thus, in any equilibrium, at most one player can play risky in some right-neighbourhood of \bar{p}_1 . Now, suppose that player 2 is the only player to experiment in some right-neighbourhood of \bar{p}_1 . Then, the relevant ODE (Equation 15 in Appendix A.2) gives us that $\lambda_2 \bar{p}_1 (1 - \bar{p}_1) v_2'(\bar{p}_1+) = \bar{p}_1 \lambda_2 (g_2 - s) - rc_2(\bar{p}_1) < 0$, as $\bar{p}_1 < \bar{p}_2$. Thus, player 2's value function drops below s immediately to the right of \bar{p}_1 , which contradicts his playing a best response. We can thus conclude that there exists some belief $\hat{p}_1 > \bar{p}_1$ such that, on (\bar{p}_1, \hat{p}_1) , player 2 plays safe. As either player can always guarantee himself his single-agent payoff by ignoring the information he gets for free from the other player, his payoff in any equilibrium is bounded below by his single-agent payoff. Thus, in any equilibrium, $v_1 > s$ on $(\bar{p}_1, \hat{p}_1]$, and player 1 experiments, while player 2 free-rides, in this range.

Thus, for beliefs right above \bar{p}_1 , in any equilibrium, player 1's payoff is given by

$$\bar{v}_1(p) = g_1 p + \bar{C}_1 u_1(p), \quad (6)$$

with $\bar{C}_1 = \frac{s - g_1 \bar{p}_1}{u_1(\bar{p}_1)}$. Player 2's equilibrium payoff for these beliefs is given by

$$\bar{v}_2(p) = s + \frac{(g_2 - s)p}{1 + r} + \bar{C}_2 u_1(p) \quad (7)$$

with $\bar{C}_2 = -\frac{(g_2 - s)\bar{p}_1}{(1 + r)u_1(\bar{p}_1)}$.

Since $\bar{C}_1 > 0$ and $\bar{C}_2 < 0$, \bar{v}_1 is strictly convex and \bar{v}_2 is strictly concave.³ The following lemma shows that the functions \bar{v}_i intersect the corresponding diagonals D_i at a unique belief.

Lemma 1 *There exists a unique $p'_1 \in (\bar{p}_1, 1)$ such that $\bar{v}_1(p'_1) = D_1(p'_1)$, and a unique $p'_2 \in (\bar{p}_2, \frac{s}{g_2})$ such that $\bar{v}_2(p'_2) = D_2(p'_2)$.*

³ \bar{v}_1 and \bar{v}_2 are obtained from Equations 16 and 18 respectively by imposing the condition $\bar{v}_i(\bar{p}_1) = s$ ($i = 1, 2$).

Proof. Please refer to Appendix B.2. ■ In the following proposition, we will show that there exists an equilibrium in cutoff strategies if and only if the degree of asymmetry between the players is high enough.

Proposition 3 *There exists a $\lambda_2^* \in (\frac{s}{h}, 1)$ such that there exists an equilibrium in cutoff strategies if and only if $\lambda_2 \in (\frac{s}{h}, \lambda_2^*]$. In this equilibrium, player 1 plays risky on $(\bar{p}_1, 1]$ and safe otherwise, while Player 2 plays risky on $(p_2', 1]$ and safe otherwise.*

Proof. Please refer to Appendix B.3. ■ Appendix B.4 shows that the belief p_2' where player 2 switches to the safe arm in the above equilibrium is strictly greater than p_2^* , the threshold in the planner's solution. This shows that for $p \in (p_2^*, p_2']$, player 2 inefficiently free-rides.

The equilibrium in cutoff strategies is depicted in Figure 2. In this equilibrium, both players' payoffs are equal to s for $p \leq \bar{p}_1$. For $p \in (\bar{p}_1, p_2']$, player i 's ($i = 1, 2$) payoff is $\bar{v}_i(p)$. For $p > p_2'$, player i 's payoff is given by

$$v_i^r(p) = g_i p + C_i^r u_0(p)$$

with $C_i^r = \frac{\bar{v}_i(p_2') - g_i p_2'}{u_0(p_2')}$.⁴ Player 1's equilibrium payoff function (v_1 , represented by the brown curve in figure 2) is (strictly) convex (on $(\bar{p}_1, 1)$); it is smooth, except for a kink at p_2' . Player 2's payoff function (v_2 , represented by the black curve in figure 2) is strictly concave on (\bar{p}_1, p_2') and strictly convex on $(p_2', 1)$; it has an inflection point at p_2' . It is smooth except for a kink at \bar{p}_1 .

Experimentation decisions are strategic substitutes. Therefore in any equilibrium, at the lowest belief where some experimentation takes place, one *pioneer* is indifferent between choosing the safe and the risky arm, given that the other player is free-riding. The *free-rider* can determine a threshold belief p_2' where he is indifferent between choosing the safe arm and the risky arm, given that the pioneer is choosing the risky arm for all beliefs between the lowest cutoff and p_2' . This implies that for beliefs just above p_2' , the free-rider finds it beneficial to experiment irrespectively of the action of the pioneer. When players are homogeneous, their free-riding opportunities are the same. At p_2' , the pioneer's payoff is less than that of the free-rider as experimentation is costly. Thus, for beliefs just above p_2' , the pioneer has an incentive to free-ride, given that the free-rider experiments. This explains (as shown in [Keller, Rady, Cripps (2005)]) why there does not exist an equilibrium where both players use cutoff strategies. However, the free-riding opportunities are different for heterogeneous players. As explained above, in any equilibrium the pioneer is always the player with the higher productivity (player 1). The lower player 2's productivity, the less player 1 has an incentive to free-ride on player 2's experimentation. If player 2's productivity is very low, player 1 no longer has

⁴These payoffs are obtained from 14 by imposing the condition $v_i^r(p_2') = \bar{v}_i(p_2')$ ($i = 1, 2$).

any incentive to free-ride on 2's experimentation for beliefs right above p'_2 . This intuitively explains the result of Proposition 3.

Geometrically, the diagonals D_1 and D_2 in Figure 2 do not coincide when players are asymmetric. As the proof of Proposition 3 shows, the condition for existence of an equilibrium in cutoff strategies is precisely that player 2 will enter the region in which risky is dominant at a more optimistic belief than player 1.⁵ This is possible if and only if the region in which risky is dominant for player 2 is relatively small enough compared to that of player 1, i.e. if and only if λ_2 is small enough compared to $\lambda_1 = 1$.

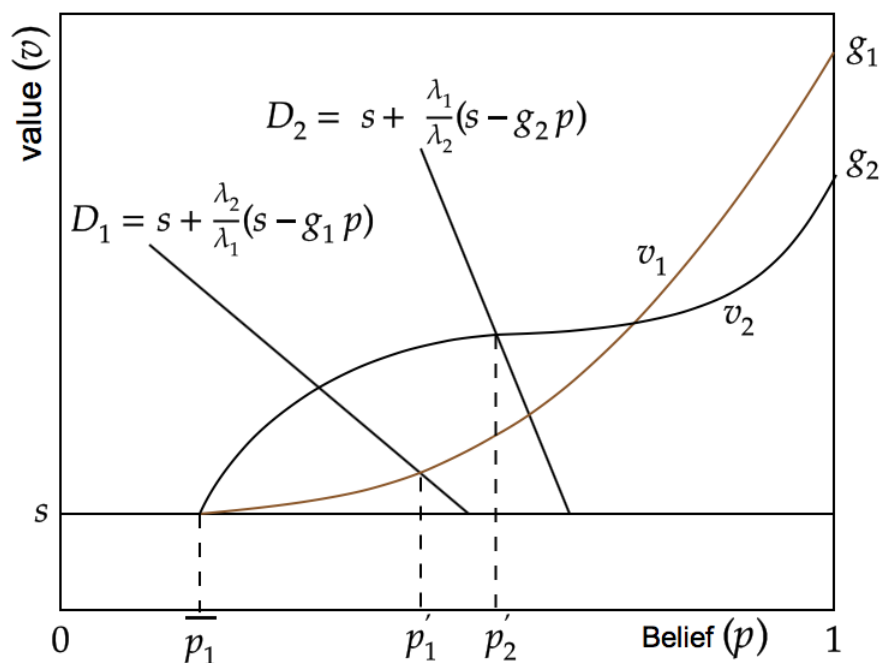


Figure 2

In Subsection 4.4, we show that if the players' learning speeds are different while the expected flow payoff from the good risky arm is the same, there again exists an equilibrium in cutoff strategies if and only if the difference in the learning speeds is high enough. The same qualitative result obtains for identical learning speeds but different expected payoffs from the good risky arm. Indeed, either form of asymmetry creates differences in the players' free-riding incentives. Diagrammatically, this can be seen by a gap between the best-response diagonals.

⁵All our figures correspond to parametric values such that the point of intersection of v_2 and v_1 lies to the right of p'_2 . However, for very low values of λ_2 , this intersection will occur to the left of p'_2 . All of our analysis goes through unchanged for this case as well.

4.2 Equilibria in non-cutoff strategies

In the previous subsection, we have identified a necessary and sufficient condition for the existence of an equilibrium in cutoff strategies. In this subsection, we will analyse equilibria where at least one of the players uses a non-cutoff strategy. To begin with, we show that even for low degrees of asymmetry, there exists an equilibrium where player 2 uses a cutoff strategy.

Proposition 4 *There exists an equilibrium in which only player 2 uses a cutoff strategy if and only if $\lambda_2 > \lambda_2^*$. In this equilibrium, the cutoff for player 2's strategy is p'_2 . Player 1 plays risky on $(\bar{p}_1, p'_2] \cup (\hat{p}_S^1, 1]$ and safe otherwise, where $\hat{p}_S^1 > p'_2$ is the belief at which player 1's payoff function and D_1 intersect.*

Proof. Please refer to Appendix B.5. ■

The equilibrium where only player 2 uses a cutoff strategy is depicted in figure 3. The brown and the black curves depict the payoffs to player 1 and 2 respectively. As the degree of asymmetry between the players is low, $p'_1 > p'_2$ and hence an equilibrium where both players use cutoff strategies does not exist. Propositions 3 and 4 together imply that there always exists an equilibrium where player 2 uses a cutoff strategy with p'_2 as the cutoff. Indeed, as argued in the previous subsection, in any equilibrium, \bar{p}_1 is the lowest belief where some experimentation takes place and only player 1 experiments at beliefs just above \bar{p}_1 . By the same token, risky is Player 2's best reply at all beliefs above p'_2 , given Player 1 plays risky on $(\bar{p}_1, p'_2]$.

When the degree of asymmetry is low, there will exist a range of beliefs just above p'_2 where player 1 free-rides. Thus, player 1 uses a non-cutoff strategy. This explains the result of Proposition 4. In the limit $\lambda_2 \downarrow \lambda_2^*$, the range above p'_2 where player 1 free-rides vanishes and hence, the equilibrium described in Proposition 4 coincides with the equilibrium in cutoff strategies.

Equilibria where at least one player uses a non-cutoff strategy always exists, as the following proposition shows. The following proposition, together with Proposition 3, fully characterises the set of all Markov perfect equilibria. To state the proposition, we let v_i be player i 's equilibrium payoff. For both players $n \in \{1, 2\}$, we define p_S^n as the (unique) point of intersection of v_n and D_n .⁶ Let $p_S^i = \min\{p_S^1, p_S^2\}$ and $p_S^j = \max\{p_S^1, p_S^2\}$.

Proposition 5 *For any $\lambda_2 \in (\frac{s}{h}, 1)$, there exists a continuum of Markov perfect equilibria in which at least one player uses a non-cutoff strategy. For each integer $l > 1$ and each sequence of threshold beliefs $(\tilde{p}_i)_{i=1}^l$ such that $\bar{p}_1 < \tilde{p}_1 < \dots < \tilde{p}_l = p_S^i$, there exists an equilibrium such that both players play safe at all beliefs $p \leq \bar{p}_1$; player 1 plays risky and player 2 plays safe in $(\bar{p}_1, \tilde{p}_1] \cup \bigcup_{i \in 2\mathbb{N} \wedge i < l} (\tilde{p}_i, \tilde{p}_{i+1}]$*

⁶The uniqueness of $p_S^n \in (\bar{p}_1, \frac{s}{g_n})$ follows from (13).

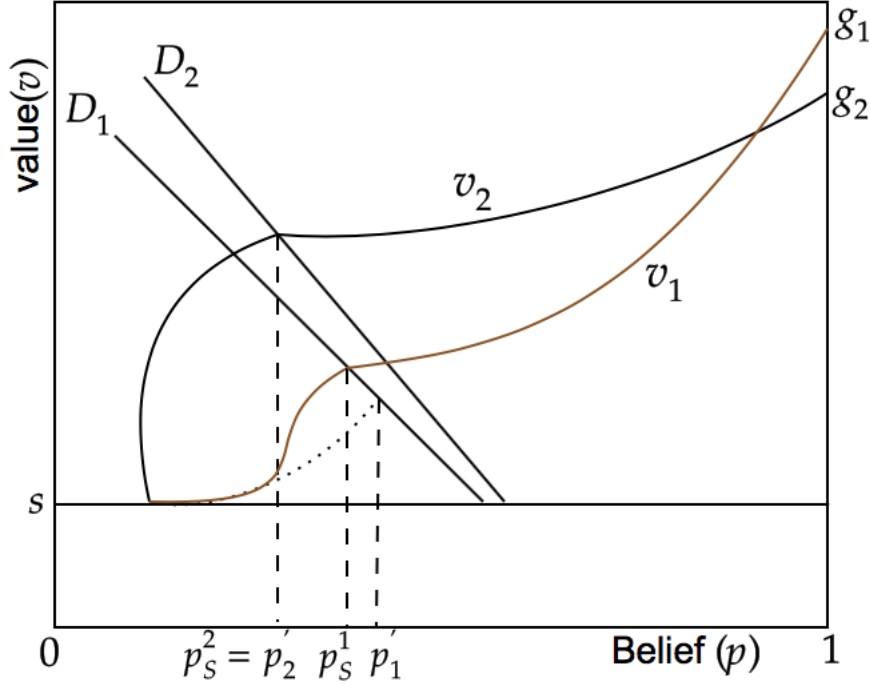


Figure 3

, while player 1 plays safe and player 2 plays risky in $\cup_{i \in 2\mathbb{N} \wedge i \leq l} (\tilde{p}_{i-1}, \tilde{p}_i]$; on $(p_S^i, p_S^j]$, player i plays risky and player j plays safe, while both players play risky on $(p_S^j, 1]$. The same strategies with $l = 1$ also describe an equilibrium in which only player 2 uses a cutoff strategy if and only if $p_2^1 = p_S^2 < p_S^1 = \hat{p}_S^1$.

On $[0, \bar{p}_1]$, both players' value function is s . For even $i < l$, on $(\tilde{p}_i, \tilde{p}_{i+1}]$, player 1's (2's) value function is given by (16) ((18)), while on $(\tilde{p}_{i-1}, \tilde{p}_i]$, player 2's (1's) value function is given by (16) ((18)); on $(p_S^i, p_S^j]$, player i 's (j 's) payoff is given by (16) ((18)). On $(p_S^j, 1]$, both players' payoffs are given by (14). The constants of integration are determined by value matching.

Proof. That the proposed strategies are mutually best responses immediately follows from our discussion at the top of Section 4. That such equilibria always exist follows immediately from the continuity of players' payoff functions and the fact that $D_i(\bar{p}_1) > s$ for both $i \in \{1, 2\}$. ■

When the degree of asymmetry is low, it is easy to observe that both players have incentives for free-riding just below p_2^1 ; i.e. safe and risky are mutually best responses in this region. Although an increase in the degree of asymmetry reduces the free-riding incentives for player 1, they never vanish completely. Therefore, there will always be a range just above \bar{p}_1 where safe and risky are mutually best responses. Hence, equilibrium allows players to take turns in experimenting at arbitrary beliefs

in (\bar{p}_1, p_2') . This explains the result of Proposition (5).

As $\bar{p}_1 < \bar{p}_2$, the proposition implies that there exist equilibria in which player 2 experiments below his single-agent threshold \bar{p}_2 . Indeed, by being the last player to experiment on $(\bar{p}_1, \tilde{p}_1]$, player 1 provides an *encouragement effect* to player 2, as the latter is willing to play risky on $(\tilde{p}_1, \tilde{p}_2]$ only because he knows that, should his experimentation not be successful, he will get to free-ride on player 1's experimentation once the belief will have dropped to \tilde{p}_1 .

4.3 Welfare rankings of equilibria

We define the *experimentation intensity* at time $t \geq 0$ as $K_t = \lambda_1 k_{1,t} + \lambda_2 k_{2,t}$, and the integral $\int_0^T K_t dt$ as the *amount of experimentation* up to time T . Thus, our measure takes into account that it matters for the information-production process which player invests time in the risky arm. Indeed, this integral measures the *output* of players' experimentation efforts. The corresponding concept in [Keller, Rady, Cripps (2005)], by contrast, rather measures the *input*, i.e. the overall resources spent on producing information, which, in the case of homogeneous players with productivities λ , of course corresponds to $1/\lambda$ times the information produced.

Lemma 2 *Suppose there is no success on the risky arm. Then, the amount of experimentation is the same in any Markov perfect equilibrium.*

Proof. As we have seen from our characterisation of equilibria, experimentation stops at \bar{p}_1 in any equilibrium. By Bayes' rule, the law of motion of the belief conditional on no success is given by $dp_t = -K_t p_t (1 - p_t) dt$. Thus, conditionally on no success, the amount of experimentation in any Markov perfect equilibrium is given by

$$\int_0^T K_t dt = \int_{p_0}^{\bar{p}_1} -\frac{dp_t}{p_t(1-p_t)} = \left[\ln \left(\frac{1-p}{p} \right) \right]_{p_0}^{\bar{p}_1},$$

which concludes the proof.

■

In the following proposition, we establish that in any equilibrium in which players swap the roles of pioneer and free-rider at least once, player 1's (2's) payoff will hit D_1 (D_2) at a more pessimistic (optimistic) belief than in the equilibrium in cutoff strategies.

Proposition 6 *Consider any equilibrium described in Proposition 5. Suppose $p_S^1 > \bar{p}_1$ is the belief at which the equilibrium payoff of player 1 meets the line D_1 and $p_S^2 > \bar{p}_1$ be the belief at which the*

equilibrium payoff of player 2 meets the line D_2 . Then, we have $p_S^1 < p_1'$. For $l > 1$ we have $p_S^2 > p_2'$ and for $l = 1$, $p_S^2 = p_2'$.

Proof. Please refer to Appendix B.6. ■

In the equilibrium in cutoff strategies, player 2 free-rides for all beliefs in $(\bar{p}_1, p_2']$. However, in any other equilibrium there exists some open subset of (\bar{p}_1, p_2') where he experiments and player 1 free-rides. Thus, for all $p \in (\bar{p}_1, p_2']$, the equilibrium in cutoff strategies gives the highest payoff to player 2, as he can free-ride on the more productive player's experimentation. This implies that, in the range $p \in (\bar{p}_1, p_2']$, player 2's payoff function in any non-cutoff equilibrium lies below his payoff in the cutoff equilibrium, and will therefore intersect the diagonal D_2 at a belief higher than p_2' . This explains why we have $p_S^2 > p_2'$. On the other hand, for all $p \in (\bar{p}_1, p_1']$, player 1 experiments in the equilibrium in cutoff strategies and receives his single-agent payoff. In any other equilibrium, however, there exists some open subset of (\bar{p}_1, p_1') where his single-agent optimal action is not a best response, and his equilibrium payoff is therefore higher. Thus, as player 1's payoff is lowest in the equilibrium in cutoff strategies, we have $p_S^1 < p_1'$.

Suppose $\lambda_2 \in (\frac{s}{h}, \lambda_2^*]$. This implies that the equilibrium in cutoff strategies exists. In the following proposition, we show that the equilibrium in cutoff strategies strictly welfare dominates all other equilibria.

Proposition 7 *Suppose $\lambda_2 \leq \lambda_2^*$ and let v_{agg}^c be the aggregate equilibrium payoff in the equilibrium in cutoff strategies and v_{agg}^{nc} be the aggregate equilibrium payoff in an arbitrary equilibrium in non-cutoff strategies. Then, $v_{agg}^c \geq v_{agg}^{nc}$, with the inequality strict on $(\bar{p}_1, 1)$.*

Proof. Please refer to Appendix (B.7). ■

First, observe that in the equilibrium in cutoff strategies, both players experiment for beliefs greater than p_2' . Since $p_S^2 > p_2'$ (by Proposition 6), the range of beliefs where both players experiment is largest in the equilibrium in cutoff strategies. Next, in the equilibrium in cutoff strategies, whenever only one player experiments, it is the player with the higher payoff arrival rate, player 1. In any other equilibrium, however, there is a range of beliefs where player 2 plays the role of the lonely pioneer. Since player 1 is more productive and in any equilibrium all experimentation ceases at \bar{p}_1 , information is most efficiently generated in the equilibrium in cutoff strategies. This intuitively explains the result of Proposition 7. One can observe that since the intensity of experimentation is highest in the equilibrium in cutoff strategies, information generation is fastest. Thus, this equilibrium involves least *delay*. This feature along with Lemma 2 also explains why the cutoff equilibrium welfare dominates all other equilibria.⁷

⁷[Dong (2018)] shows that if the players' initial beliefs are asymmetric enough, equilibrium welfare improves.

The comparison between the equilibrium in cutoff strategies and an equilibrium in which players swap roles once is depicted in Figure 4(a).⁸ Figures 4(b) and 4(c) depict the actions of players in the equilibrium in cutoff strategies and the equilibrium in non-cutoff strategies respectively. These equilibria correspond to the ones depicted in Figure 4(a).

The brown curve (v_1) and the black curve (v_2) in Figure 4 (a) depict the payoffs to player 1 and 2 respectively in the equilibrium in cutoff strategies. In the equilibrium in non-cutoff strategies, payoffs coincide for beliefs less than or equal to \tilde{p}_1 . At \tilde{p}_1 , players switch arms. The dotted blue curve depicts the payoff to player 1 and the dotted red curve depicts the payoff to player 2 in the equilibrium in non-cutoff strategies for $p > \tilde{p}_1$. As argued, the blue curve meets the line D_1 at a belief p_S^1 , which is strictly less than p_1' . In the region $(\tilde{p}_1, p_S^1]$, player 2 experiments and player 1 free rides. At p_S^1 , player 1 switches to the risky arm and player 2 switches to the safe arm. When the red curve meets the line D_2 at $p_S^2 > p_2'$, player 2 switches to the risky arm again.

When $\lambda_2 > \lambda_2^*$, the equilibrium in cutoff strategies does not exist. However, the argument in the proof of Proposition 7 allows us to show that, on $(\bar{p}_1, p_2']$, the equilibrium of Proposition 4, which is the only equilibrium in which player 1 is experimenting throughout this range, strictly welfare-dominates all other equilibria. Indeed, with heterogeneous players, more frequent switches have the effect of replacing experimentation by the strong player with experimentation by the weak player in some open subset in (\bar{p}_1, p_2') . Thus, even though more frequent switches can expand the range of beliefs where both experiment, there is always a welfare loss in the range $(\bar{p}_1, p_2']$. Hence, if players switch the role of pioneer and free-rider more frequently, the equilibrium welfare is not unambiguously improved. This is in contrast to the case with homogeneous players ([Keller, Rady, Cripps (2005)]), where the only effect of increasing the frequency of switches is to expand the range of beliefs where both players experiment, thus unambiguously improving equilibrium welfare. Yet, we have not been able to establish that the equilibrium of Proposition 4 is *globally* welfare-maximising.

4.4 Learning rates versus payoffs

In our baseline model, we have considered asymmetric Poisson arrival rates only. However, since the expected lumpsum payoff from the good risky arm was the same for both players, the asymmetry in learning rates implied that the expected flow payoff from a good risky arm was also different across the players. In this subsection, we will analyse a model where learning rates differ but the expected

⁸Proposition 6 implies that the qualitative characteristics of p_s^1 and p_s^2 are the same in any equilibrium in non-cutoff strategies. For simplicity, we consider an equilibrium in non-cutoff strategies where players swap roles only once in the figure.

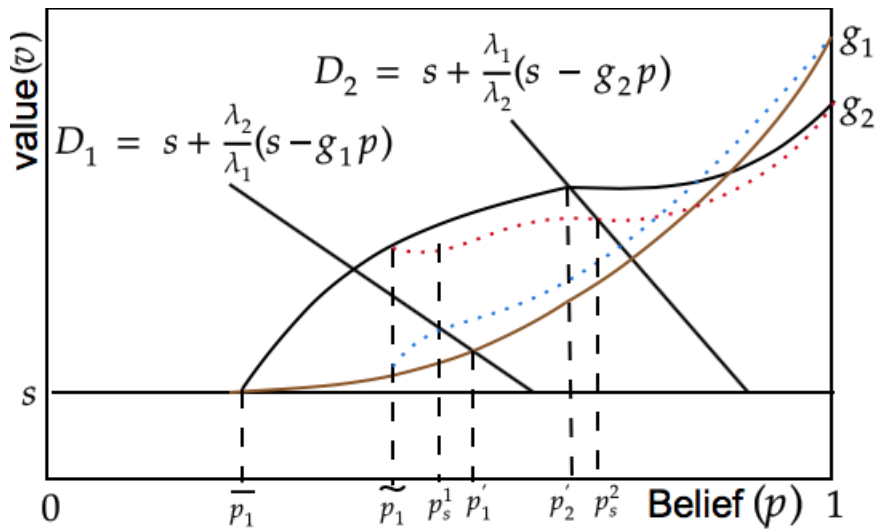


Figure 4(a)

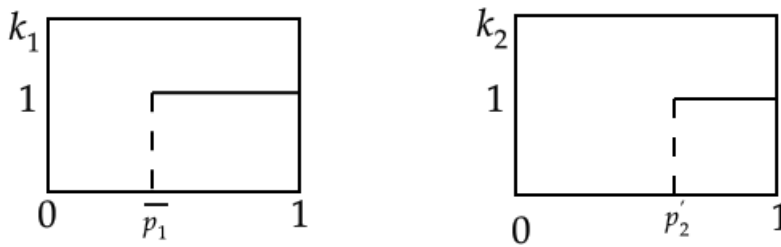


Figure 4(b): Actions of players in the equilibrium in cutoff strategies

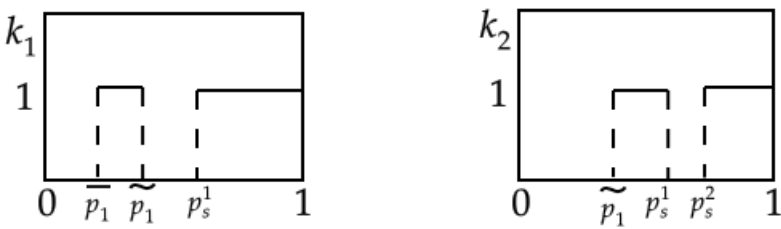


Figure 4(c): Actions of players in the equilibrium in non-cutoff strategies

flow payoff from a good risky arm is the same for both players.

Define $\hat{g} = \lambda_1 h_1$ where $\lambda_1 = 1$ and $h_1 > 0$. For any $\lambda_2 \in (0, 1)$, we choose a $h_2 > 0$ such that $\lambda_2 h_2 = \hat{g}$.

We will first analyse the social planner's problem. The planner's value function w satisfies the Bellman equation

$$w(p) = 2s + \max_{k_1, k_2 \in \{0, 1\}} \{k_1[\bar{B}_1(p, w) - \bar{c}(p)] + k_2[\bar{B}_2(p, w) - \bar{c}(p)]\} \quad (8)$$

where

$$\bar{B}_i(p, w) = \frac{\lambda_i p [2\hat{g} - w(p) - w'(p)(1-p)]}{r}$$

and $\bar{c}(p) = s - \hat{g}p$. The following proposition will show that the structure of the planner's solution is the same as in Proposition 1.

Proposition 8 *The planner's optimal policy $k^*(p) = (k_1^*, k_2^*)(p)$ is given by*

$$(k_1^*, k_2^*)(p) = \begin{cases} (1, 1) & \text{if } p \in (\bar{p}_2^*, 1) \\ (1, 0) & \text{if } p \in (\bar{p}_1^*, \bar{p}_2^*] \\ (0, 0) & \text{if } p \in (0, \bar{p}_1^*] \end{cases}$$

and the value function is

$$w(p) = \begin{cases} 2\hat{g}p + \left[\frac{\lambda}{\lambda_2} s - \hat{g}\bar{p}_2^* \frac{1-\lambda_2}{\lambda_2} - 2\hat{g}\bar{p}_2^* \right] \frac{u_0(p)}{u_0(\bar{p}_2^*)} & \text{if } p \in (\bar{p}_2^*, 1], \\ s + \left[\frac{2\hat{g}+r\hat{g}}{1+r} - \frac{s}{1+r} \right] p + \left[s - \left(\frac{2\hat{g}+r\hat{g}}{1+r} - \frac{s}{1+r} \right) \bar{p}_1^* \right] \frac{u_1(p)}{u_1(\bar{p}_1^*)} & \text{if } p \in (\bar{p}_1^*, \bar{p}_2^*], \\ 2s & \text{if } p \in (0, \bar{p}_1^*], \end{cases}$$

where \bar{p}_1^* is defined as

$$\bar{p}_1^* = \frac{rs}{2(\hat{g} - s) + r\hat{g}}, \quad (9)$$

and $\bar{p}_2^* \in (\bar{p}_1^*, \frac{s}{\hat{g}})$ is implicitly defined by $w(\bar{p}_2^*) = \frac{\lambda}{\lambda_2} s - \hat{g}\bar{p}_2^* \frac{1-\lambda_2}{\lambda_2}$.

Proof.

Proof is by a standard verification argument. Please see the Appendix B.8 for details.

■

We will now analyse the non-cooperative game. Player i 's ($i = 1, 2$), value function w_i satisfies the Bellman equation

$$w_i(p) = s + k_j(p)\lambda_j\bar{b}_i(p, w_i) + \max_{k_i \in \{0,1\}} k_i[\lambda_i\bar{b}_i(p, w_i) - (s - \hat{g}p)] \quad (10)$$

where

$$\bar{b}_i(p, w) = p \frac{\hat{g} - w_i(p) - (1-p)w_i'(p)}{r}$$

As before, we can derive the best response diagonals as

$$\bar{D}_i(p) = s + \frac{\lambda_j}{\lambda_i}[s - \hat{g}p]$$

The single agent thresholds are $\hat{p}_i = \frac{rs}{rs+(r+\lambda_i)(\hat{g}-s)}$. It can be verified that $\hat{p}_1 < \hat{p}_2$. As in the baseline model, we can argue that in any equilibrium, \hat{p}_1 is the lowest belief where some experimentation takes place and player 1 is the last one to experiment. This implies that, in any equilibrium, for beliefs right above \hat{p}_1 , payoffs to player 1 and 2 are given by $\bar{w}_1(p)$ and $\bar{w}_2(p)$ respectively such that

$$\bar{w}_1(p) = \hat{g}p + [s - \hat{g}\hat{p}_1] \frac{u_1(p)}{u_1(\hat{p}_1)}$$

$$\bar{w}_2(p) = s + \frac{\hat{g}-s}{1+r}p - \frac{\hat{g}-s}{1+r}\hat{p}_1 \frac{u_1(p)}{u_1(\hat{p}_1)}$$

It can be verified that \bar{w}_1 is strictly convex and \bar{w}_2 is strictly concave. By arguments similar to those in Lemma 1, we can infer that there exists a unique $\bar{p}_1 \in (\hat{p}_1, 1)$ such that $\bar{w}_1(\bar{p}_1) = D_1(\bar{p}_1)$ and a unique $\bar{p}_2 \in (\hat{p}_2, \frac{s}{\hat{g}})$ such that $\bar{w}_2(\bar{p}_2) = D_2(\bar{p}_2)$. In the following proposition, we establish that an equilibrium in cutoff strategies exists if and only if the degree of asymmetry is high enough.

Proposition 9 *There exists a $\hat{\lambda}_2 \in (0, 1)$ such that there exists an equilibrium in cutoff strategies if and only if $\lambda_2 \in (0, \hat{\lambda}_2]$. In this equilibrium, player 1 plays risky on $(\hat{p}_1, 1]$ and safe otherwise, while player 2 plays risky on $(\bar{p}_2, 1]$ and safe otherwise.*

Proof. Please refer to Appendix B.9 for details. ■ Figure 5 depicts the equilibrium in cutoff strategies. The brown (black) curve depicts the payoffs to player 1 (2). Since the flow payoff obtained by each player from a good risky arm is fixed at \hat{g} , the point of intersection between the best-response line and the horizontal line $w = s$ is the same for both players. As agents become more asymmetric, the best response lines diverge more from each other. Due to this, there emerges a range of beliefs where only player 2 can free ride. Hence, if the degree of asymmetry between the players is high enough, there exists an equilibrium in cutoff strategies.

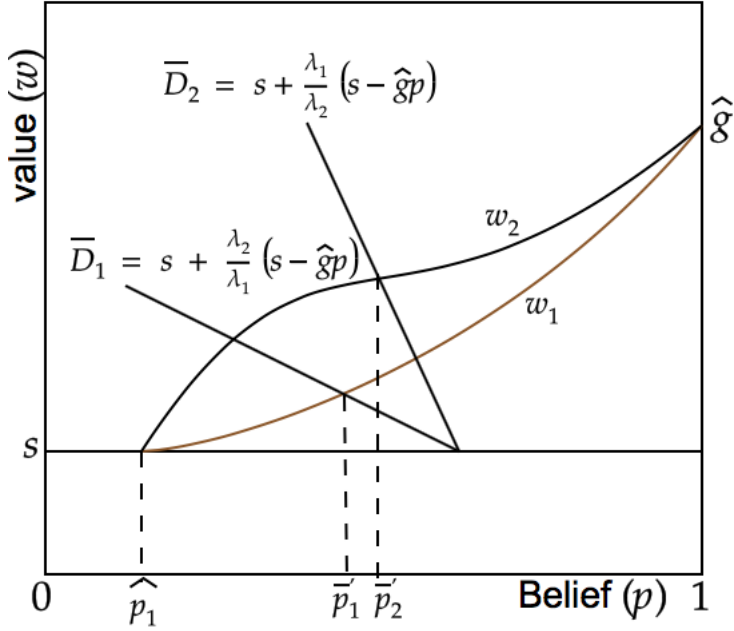


Figure 5

Using similar arguments, we can establish that when the players' learning rates are equal but their flow payoffs from a good risky arm are different, an equilibrium in cutoff strategies exists if the asymmetry between the players is high enough. As an illustration, suppose $\lambda_1 = \lambda_2 = \hat{\lambda}$. The lump sum received by each player from a good risky arm at the jumping times of the Poisson process with intensity $\hat{\lambda}$ is drawn from a time-invariant distribution. The mean of this distribution h_i ($i = 1, 2$) is such that $h_1 > h_2$ and $h_2 \geq \frac{s}{\hat{\lambda}}$. This implies $g_1 > g_2 \geq s$. The best response diagonal of player i ($i = 1, 2$) is now given by $\hat{D}_i : v = 2s - g_i p$. Figure 6 shows an equilibrium in cutoff strategies in this framework. The red (blue) line depicts the diagonal \hat{D}_1 (\hat{D}_2) and the brown (black) curve depicts the payoffs to player 1 (2). This equilibrium exists only when $\tilde{p}_2 > \tilde{p}_1$, which happens when the players are highly asymmetric and the best response diagonals are far apart from each other.

5 Conclusion

In this paper, we have characterised the set of Markov perfect equilibria in a two-armed bandit model with heterogeneous players. We have shown that there always exists an equilibrium in which the weaker player uses a cutoff strategy. If the heterogeneity is stark enough, there exists an equilibrium

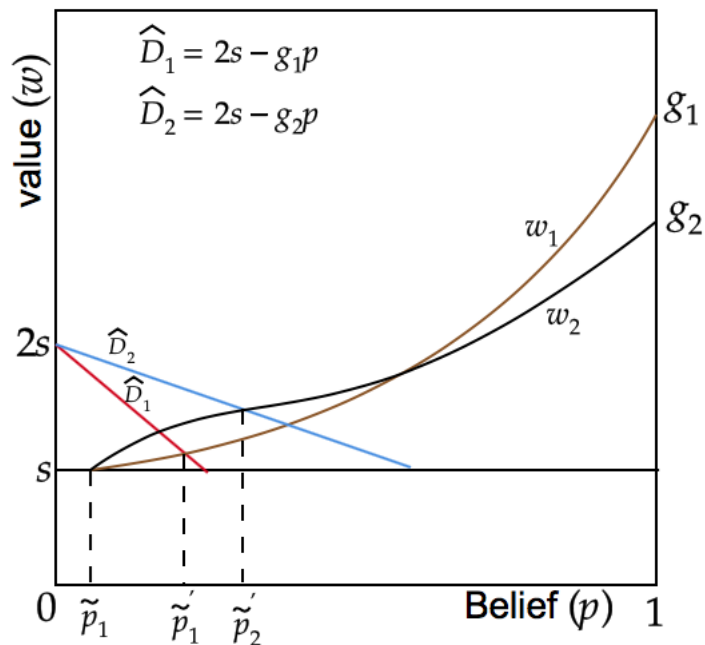


Figure 6

in cutoff strategies. If such an equilibrium exists, it is welfare-optimal.

Thus, suppose there are two oil companies with vastly different drilling technologies, e.g. a big multinational firm and a small local enterprise. One could argue that the difference in technological capabilities between the two will be bigger in developing countries. On account of the big heterogeneity in capabilities, we should expect the equilibrium in cutoff strategies to exist. An empirically testable prediction of our model would thus be that there will be a higher frequency of instances in developing countries where the small local firm would free-ride on the experimentation provided by the big multinational firm, and only enter the market after oil had been struck, even if the original level of uncertainty regarding the presence of oil was only moderate.

We have restricted players to using one arm only at any given instant t . By the linearity of the players' Bellman equations, our equilibria would remain equilibria if we allowed players to select experimentation intensities $k_{i,t} \in [0, 1]$. There might, however, be more equilibria in this case.

Our analysis has relied heavily on the characterisation of players' best responses via the diagonals D_i (see Equation (4)), which was pioneered by [Keller, Rady, Cripps (2005)] for the homogeneous-player case. We expect that a similar approach could, *mutatis mutandis*, be used to study other kinds of asymmetries, e.g. pertaining to players' safe-arm payoffs s_i . We should expect a similar result to

our Proposition 3 to hold in these cases, namely that there existed an equilibrium in cutoff strategies if and only if the heterogeneity was stark enough.

References

- [Besanko and Wu (2013)] Besanko, D., Wu, J., 2013: “The impact of market structure and learning on the tradeoff between R & D competition and cooperation”, *The Journal of Industrial Economics* 61, 166 – 201
- [Bolton and Harris (1999)] Bolton, P., Harris, C., 1999: “Strategic Experimentation”, *Econometrica* 67, 349 – 374.
- [Bonatti and Hörner (2011)] Bonatti, A., Hörner, J., 2011: “Collaborating”, *American Economic Review* 101, 632 – 663
- [Dong (2018)] Dong, M., 2018: “Strategic Experimentation with Asymmetric Information”, *mimeo, Penn State University*
- [Heidhues, Rady, Strack (2015)] Heidhues, P., Rady, S., Strack, P., 2015: “Strategic experimentation with private payoffs”, *Journal of Economic Theory* 159, 531 – 551
- [Hörner and Skrzypacz (2016)] Hörner, J., Skrzypacz, A., 2016: “Learning, Experimentation and Information Design”, *Working paper, Stanford University*.
- [Keller, Rady, Cripps (2005)] Keller, G., Rady, S., Cripps, M., 2005: “Strategic Experimentation with Exponential Bandits”, *Econometrica* 73, 39 – 68.
- [Keller and Rady (2010)] Keller, G., Rady, S., 2010: “Strategic Experimentation with Poisson Bandits”, *Theoretical Economics* 5, 275 – 311.
- [Klein (2013)] Klein, N., 2013: “Strategic Learning in Teams”, *Games and Economic Behavior* 82, 632 – 657.
- [Klein and Rady (2011)] Klein, N., Rady, S., 2011: “Negatively Correlated Bandits”, *The Review of Economic Studies* 78, 693 – 792.
- [Malueg and Tsutsui (1997)] Malueg, D., Tsutsui, S., 1997: “Dynamic R & D Competition with Learning”, *The RAND Journal of Economics* 28, 751 – 772

[Murto and Välimäki (2011)] Murto, P., Välimäki, J., 2011: “Learning and Information Aggregation in an Exit Game”, *Review of Economic Studies* 78, 1426 – 1461.

[Presman (1990)] Presman, E.L., 1990: “Poisson Version of the Two-Armed Bandit Problem with Discounting”, *Theory of Probability and its Applications*

[Rosenberg, Salomon, Vieille (2013)] Rosenberg, D., Salomon, A., Vieille, N., 2013: “On games of Strategic Experimentation”, *Games and Economic Behavior* 82, 31 – 51.

[Rosenberg, Solan, Vieille (2007)] Rosenberg, D., Solan, E., Vieille, N., 2007: “Social Learning in One-arm Bandit Problems”, *Econometrica* 75, 1511 – 1611.

[Thomas (2017)] Thomas, C., 2017: “Experimentation with Congestion”, *mimeo, University of Texas Austin*.

APPENDIX

A Ordinary Differential Equations

A.1 ODEs in the planner’s problem

Clearly, if $(k_1, k_2) = (0, 0)$ is played at a belief p , the planner’s payoff function satisfies $v(p) = 2s$.

If the planner plays $k_1 = k_2 = 1$ on an open set of beliefs, his payoff function on this set satisfies

$$v(p) = 2s + B_1(p, v) - c_1(p) + B_2(p, v) - c_2(p),$$

which is equivalent to the ODE

$$\lambda p(1 - p)v'(p) + (r + \lambda p)v(p) = (r + \lambda)pg. \quad (11)$$

This is solved by

$$v(p) = gp + Cu_0(p)$$

where C is a constant of integration.

By the same token, the ODE for $(k_1, k_2) = (1, 0)$ is given by

$$p(1 - p)v'(p) + (r + p)v(p) = r(s + pg_1) + pg. \quad (12)$$

This is solved by

$$v(p) = s + \left[\frac{g + rg_1}{1+r} - \frac{s}{1+r} \right] p + Cu_1(p).$$

A.2 ODEs of players in the non-cooperative game

If $k_1 = k_2 = 0$, both players' payoff functions satisfy $v_i(p) = s$.

If $k_1 = k_2 = 1$ prevails on an open set of beliefs in the non-cooperative game, both players' value function for beliefs in this set satisfies

$$\lambda p(1-p)v_i'(p) + (r + \lambda p)v_i(p) = (r + \lambda)pg_i. \quad (13)$$

This is solved by

$$v_i = g_i p + Cu_0(p) \quad (14)$$

where C is a constant of integration.

If $k_i = 1$ and $k_j = 0$, player i 's payoff function satisfies

$$\lambda_i p(1-p)v_i'(p) + (r + \lambda_i p)v_i(p) = (r + \lambda_i)pg_i. \quad (15)$$

Solving this, we get

$$v_i(p) = g_i p + Cu_i(p) \quad (16)$$

where $u_i(p) = (1-p)\left[\frac{1-p}{p}\right]^{\mu_i}$ and $\mu_i = \frac{r}{\lambda_i}$. Player j 's payoff function satisfies

$$\lambda_i p(1-p)v_j'(p) + (r + \lambda_i p)v_j(p) = rs + \lambda_i pg_j. \quad (17)$$

This is solved by

$$v_j = s + \frac{\lambda_i}{\lambda_i + r}(g_j - s)p + Cu_i(p). \quad (18)$$

B Proofs

B.1 Proof of Proposition 1

The function v satisfies $v = 2s$ on $[0, p_1^*]$, $v = 2s + B_1 - c_1$ on $(p_1^*, p_2^*]$ and $v = 2s + B_1 - c_1 + B_2 - c_2$ on $(p_2^*, 1]$;⁹ thus, v is the payoff function associated with the policy k^* .¹⁰ We shall first show that v is of class C^1 , (strictly) increasing and (strictly) convex (on $(p_1^*, 1)$).

One computes that, for $p \in (p_1^*, p_2^*)$, $B_1(p, v) - c_1(p) = \psi(p)$, where ψ is defined as

$$\psi(p) = -s + pg_1 + \frac{1}{r}p \left[g - s - \frac{g + rg_1}{1+r} + \frac{s}{1+r} + \frac{r}{p} \left(s - p_1^* \left(\frac{g + rg_1}{1+r} - \frac{s}{r+1} \right) \right) \frac{u_1(p)}{u_1(p_1^*)} \right].$$

Direct computation shows that $u_1'' > 0$ and $s - p_1^* \left(\frac{g + rg_1}{1+r} - \frac{s}{r+1} \right) > 0$, so that ψ , and hence $v|_{(p_1^*, p_2^*)}$, is strictly convex. One furthermore shows by direct computation that $\psi(p_1^*) = \psi'(p_1^*) = 0$, implying that $v|_{(0, p_2^*)}$ is of class C^1 .

We shall now show that p_2^* is well-defined. Indeed, by definition, $x = p_2^*$ must satisfy

$$\left[\frac{g + rg_1}{1+r} - \frac{s}{1+r} \right] x + \left[s - \left(\frac{g + rg_1}{1+r} - \frac{s}{1+r} \right) p_1^* \right] \frac{u_1(x)}{u_1(p_1^*)} = \frac{s}{\lambda_2}.$$

The left-hand side of this equation is strictly increasing in x for $x > p_1^*$ and equal to $s < \frac{s}{\lambda_2}$ at $x = p_1^*$. Furthermore, at $x = \frac{s}{g_2}$, the left-hand side exceeds $\left[\frac{g + rg_1}{1+r} - \frac{s}{1+r} \right] \frac{s}{g_2} > \frac{s}{\lambda_2}$. By continuity, the equation thus admits of a unique root $p_2^* \in (p_1^*, \frac{s}{g_2})$.

As $p_2^* < \frac{s}{g_2}$, $\frac{\lambda}{\lambda_2}s - p_2^*g > 0$, and $v|_{[p_2^*, 1]}$ is strictly convex as well. It remains to show that $v|_{[p_2^*, 1]}$ is also strictly increasing. By convexity, it is sufficient to show smooth pasting at p_2^* . By the ODE for the region (p_1^*, p_2^*) (Equation 12 in Appendix A.1), we have $p_2^*(1 - p_2^*)v'(p_2^*-) = \left[rs + rp_2^*g_1 + p_2^*g - (r + p_2^*)\frac{\lambda}{\lambda_2}s \right]$. By the ODE for the $(p_2^*, 1)$ -region (Equation 11 in Appendix A.1), we find $p_2^*(1 - p_2^*)v'(p_2^*+) = \left[(r + \lambda)p_2^*g - (r + \lambda p_2^*)\frac{\lambda}{\lambda_2}s \right] / \lambda$, and hence $v'(p_2^*+) = v'(p_2^*-)$.

It remains to show that v solves the Bellman equation, i.e. that $B_i \leq c_i$ for both $i \in \{1, 2\}$ on $[0, p_1^*]$; $B_1 \geq c_1$ and $B_2 \leq c_2$ on $(p_1^*, p_2^*]$; and $B_i \geq c_i$ for both $i \in \{1, 2\}$ on $(p_2^*, 1]$. First, let $p \in [0, p_1^*]$. In this case, $v = 2s$, and $B_i \leq c_i$ if and only if $p \leq \frac{rs}{rg_i + \lambda_i(g - 2s)}$, which is verified for all $p \leq p_1^*$. Now, let $p \in (p_1^*, p_2^*]$. As v is strictly increasing in this range, $v = 2s + B_1 - c_1 > 2s$, and thus $B_1 > c_1$. Moreover, $v = 2s + B_1 - c_1$ implies that $B_2 = \lambda_2 B_1 = \lambda_2(v - s - pg_1) \leq s - pg_2 = c_2$ if and only if $v \leq \frac{\lambda}{\lambda_2}s$, which is verified as $p \leq p_2^*$. Finally, let $p \in (p_2^*, 1)$. In this range, we have that

⁹We suppress arguments whenever this is convenient.

¹⁰In Appendix A.1, we display the ODEs that v satisfies for each range of beliefs and the corresponding general form of v for that range. The specific value of v is obtained by value matching.

$g - v - (1 - p)v' = \frac{r}{\lambda p} \left(\frac{\lambda}{\lambda_2} s - p_2^* g \right) \frac{u_0(p)}{u_0(p_2^*)}$, so that $B_i = \frac{\lambda_i}{\lambda} v - pg_i$, which exceeds $c_i = s - pg_i$ if and only if $v \geq \frac{\lambda}{\lambda_i} s$. By monotonicity of v , $v \geq \frac{\lambda}{\lambda_2} s > \lambda s$ in this range, which completes the proof.

B.2 Proof of Lemma 1

The function \bar{v}_1 is strictly increasing, while D_1 is strictly decreasing. Furthermore, $\bar{v}_1(\bar{p}_1) < D_1(\bar{p}_1)$ and $\bar{v}_1(1) > D_1(1)$. Since both \bar{v}_1 and D_1 are moreover continuous, there exists a unique $p'_1 \in (\bar{p}_1, 1)$ such that $\bar{v}_1(p'_1) = D_1(p'_1)$.

As $\bar{C}_2 < 0$, we have

$$\bar{v}_2(\bar{p}_2) < s + \frac{[g_2 - s]\bar{p}_2}{1 + r} = s + \frac{[g_2 - s]}{1 + r} \frac{\mu_2 s}{(\mu_2 + 1)g_2 - s} \equiv \bar{\Psi}.$$

On the other hand, $D_2(\bar{p}_2) = s + \frac{s[g_2 - s]}{\lambda_2[(\mu_2 + 1)g_2 - s]}$. This implies

$$D_2(\bar{p}_2) - \bar{\Psi} = \frac{s[g_2 - s]}{[(\mu_2 + 1)g_2 - s]\lambda_2(1 + r)} > 0.$$

Hence, $D_2(\bar{p}_2) > \bar{v}_2(\bar{p}_2)$. Furthermore, the function \bar{v}_2 is strictly increasing, and D_2 is strictly decreasing, on $(\bar{p}_2, \frac{s}{g_2})$, while $\bar{v}_2(\frac{s}{g_2}) > D_2(\frac{s}{g_2}) = s$. Since both \bar{v}_2 and D_2 are moreover continuous, there exists a unique $p'_2 \in (\bar{p}_2, \frac{s}{g_2})$ such that $\bar{v}_2(p'_2) = D_2(p'_2)$.

B.3 Proof of Proposition 3

Proof. By our previous arguments, in any equilibrium in cutoff strategies, player 1 will play risky on $(\bar{p}_1, 1]$ and safe otherwise. In response, by the definition of p'_2 , player 2 must play risky on $(p'_2, 1]$ and safe otherwise, if there is an equilibrium in cutoff strategies. Indeed, below p'_2 , player 2 is playing a best response to player 1's action choice by the definition of p'_2 . Since D_2 is decreasing, it is sufficient to show that player 2's payoff function is increasing on $[p'_2, 1]$ in order to show that he is also playing a best response at beliefs above p'_2 . Firstly, we note that the closed-form expression for player 2's payoff function (see Equation 14 in Appendix A.2) implies that player 2's payoff v_2 is strictly convex on $(p'_2, 1)$, as $v_2(p'_2) = D_2(p'_2) > g_2 p'_2$, where the inequality follows from $p'_2 < \frac{s}{g_2}$ (see Lemma 1). Furthermore, the relevant ODEs (Equations 17 and 13 in Appendix A.2) show that $v_2(p'_2) = D_2(p'_2)$ implies smooth pasting at p'_2 . As moreover $\bar{v}'_2 > 0$ (as $\bar{C}_2 < 0$ and $u'_1 < 0$), we can conclude that player 2's value function is strictly increasing on $(p'_2, 1)$ as well, and hence that player 2 is playing a best response at beliefs above p'_2 .

Thus, the candidate strategy profile is indeed an equilibrium if and only if player 1's strategy is a best response to player 2's. This requires player 2 to choose the safe arm for all beliefs at which player 1's payoff is below D_1 . Thus, it remains to determine under what conditions $p'_2 \geq p'_1$.

We will first argue that p'_1 (p'_2) is increasing (decreasing) in λ_2 . Recall that p'_1 is the point of intersection of the function \bar{v}_1 and the line D_1 . As λ_2 decreases, the line D_1 rotates anticlockwise around the point $(\frac{s}{g_1}, s)$. Since \bar{v}_1 is independent of λ_2 , p'_1 decreases as λ_2 decreases. On the other hand, as λ_2 decreases, the line D_2 shifts to the right and becomes steeper. By direct computation, one shows that \bar{v}_2 becomes flatter as λ_2 decreases. This implies that p'_2 increases.

Consider the case $\lambda_2 \downarrow \frac{s}{h}$. Then, $D_2 \rightarrow s + \frac{(s-sp)}{\lambda_2}$. Thus, the belief \hat{p} such that $D_2(\hat{p}) = s$ will tend to 1. Moreover, $\bar{v}_2 \rightarrow s$. Hence, $p'_2 \rightarrow 1$. However, D_1 still intersects the line s at $p = \frac{s}{g_1}$, implying that $p'_1 \leq \frac{s}{g_1} < p'_2$.

Next, we consider the case $\lambda_2 \uparrow 1$ and argue that there exists a left neighborhood of 1 such that, for all λ_2 in this neighborhood, $p'_2 < p'_1$. Now, suppose that $\bar{v}_1(p^\dagger) = \bar{v}_2(p^\dagger)$ while $\bar{v}'_1(p^\dagger) \geq \bar{v}'_2(p^\dagger)$ for some p^\dagger in the interior of the $(1, 0)$ -region. The relevant ODEs then imply that $p^\dagger \geq \check{p} = \frac{rs}{rg_1 + (g_1 - g_2)}$. It thus follows that $\bar{v}_2 > \bar{v}_1$ for all beliefs in $(\bar{p}_1, \check{p}]$. Note that $\check{p} < \frac{s}{g_1}$ and $\check{p} \uparrow \frac{s}{g_1}$ as $\lambda_2 \uparrow 1$. Furthermore, recall that p'_1 is implicitly defined by

$$(1 + \lambda_2)(g_1 p'_1 - s) + \bar{C}_1 u_1(p'_1) = 0,$$

where we note that \bar{C}_1 and u_1 are both independent of λ_2 . This implies that (1) $p'_1 < \frac{s}{g_1}$ for all $\lambda_2 \in [\frac{s}{h}, 1]$ (as $\bar{C}_1 > 0$ and $u_1 > 0$ for $p < 1$), and (2) that p'_1 is a continuous function of λ_2 (by the Implicit Function Theorem). Therefore $\hat{p} = \max_{\lambda_2 \in [\frac{s}{h}, 1]} p'_1 < \frac{s}{g_1}$. Thus, we can choose $\underline{\lambda}_2 \in (\frac{s}{h}, 1)$ such that, for all $\lambda_2 \in [\underline{\lambda}_2, 1]$, $\check{p} > \hat{p}$, and therefore $\bar{v}_2 > \bar{v}_1$ on $(\bar{p}_1, p'_1]$. It thus follows that, for $\lambda_2 \in [\underline{\lambda}_2, 1]$, $\bar{p}_2 < p'_1$, where \bar{p}_2 is the belief where the function \bar{v}_2 intersects the line D_1 . As $p'_2 \downarrow \bar{p}_2$ for $\lambda_2 \uparrow 1$, we can conclude that there exists some $\hat{\lambda}_2 \in (\frac{s}{h}, 1)$ such that, for all $\lambda_2 \in (\hat{\lambda}_2, 1)$, $p'_2 < p'_1$. Thus, by monotonicity of p'_1 and p'_2 in λ_2 , there exists a unique $\lambda_2^* \in (\frac{s}{h}, 1)$ such that $p'_2 \geq p'_1$ if and only if $\lambda_2 \in (\frac{s}{h}, \lambda_2^*]$. ■

B.4 To show that $p_2^* < p'_2$

Recall from the proof of Proposition 9 that p_2^* was implicitly defined as the unique root of the (for $p > p_1^*$) strictly increasing function ζ , where

$$\zeta(p) = \left[g_1 + \frac{g_2 - s}{1 + r} \right] p + \left[s - \left(g_1 + \frac{g_2 - s}{1 + r} \right) p_1^* \right] \frac{u_1(p)}{u_1(p_1^*)} - \frac{s}{\lambda_2}.$$

By the same token, p'_2 is implicitly defined by $\bar{v}_2(p'_2) = D_2(p'_2)$, which is equivalent to

$$\frac{g_2 - s}{1 + r} p'_2 + p'_2 g_1 - \frac{s}{\lambda_2} = \frac{g_2 - s}{1 + r} \bar{p}_1 \frac{u_1(p'_2)}{u_1(\bar{p}_1)}.$$

As $p'_2 > \bar{p}_2 > p_1^*$, it remains to show that

$$\zeta(p'_2) = \frac{g_2 - s}{1 + r} \bar{p}_1 \frac{u_1(p'_2)}{u_1(\bar{p}_1)} + \left[s - \left(g_1 + \frac{g_2 - s}{1 + r} \right) p_1^* \right] \frac{u_1(p'_2)}{u_1(p_1^*)} > 0.$$

For this, it is sufficient that

$$s - \left(g_1 + \frac{g_2 - s}{1 + r} \right) p_1^* > 0,$$

which follows by direct computation.

B.5 Proof of Proposition 4

If $\lambda_2 \leq \lambda_2^*$, $p'_1 \leq p'_2$, by the proof of Proposition 3. Suppose to the contrary that the equilibrium in which only player 2 uses a cutoff exists. By Proposition 6, $p_1^S < p'_1 \leq p'_2 = p_2^S$, a contradiction to the characterisation of this equilibrium in Proposition 5.

Now, suppose $\lambda_2 > \lambda_2^*$. By the proof of Proposition 3, $p'_1 > p'_2$. It thus remains to show that $\hat{p}_1^S > p'_2$. Yet, player 1's payoff from the conjectured equilibrium strategies at p'_2 is given by $\bar{v}_1(p'_2) < D_1(p'_2)$, the inequality being immediately implied by $p'_1 > p'_2$, we have $\hat{p}_1^S > p'_2$, and, by Proposition 5, the equilibrium exists.

B.6 Proof of Proposition 6

First, consider $l > 1$. It is sufficient to show that $\bar{v}_1 < v_1$ and $\bar{v}_2 > v_2$ on $(\tilde{p}_1, p_1^S]$, where v_n is player n 's equilibrium payoff function.

Note that $\bar{v}_n(\tilde{p}_1) = v_n(\tilde{p}_1)$ for both $n \in \{1, 2\}$ and suppose that $\bar{v}_2(\tilde{p}_{i-1}) \geq v_2(\tilde{p}_{i-1})$ and $\bar{v}_1(\tilde{p}_{i-1}) \leq v_1(\tilde{p}_{i-1})$ for some $i \in \{2, \dots, k\}$. Suppose that $i - 1 \geq 1$ is odd, and let v_1^{rr} be player 1's payoff from deviating to playing risky on $(\tilde{p}_{i-1}, \tilde{p}_i]$. By construction, $v_1^{rr}(\tilde{p}_{i-1}) = v_1(\tilde{p}_{i-1}) \geq \bar{v}_1(\tilde{p}_{i-1})$. Suppose to the contrary that there exists a belief $p \in (\tilde{p}_{i-1}, \tilde{p}_i]$ such that $\bar{v}_1(p) = v_1^{rr}(p)$. The relevant ODEs ((15) and (13)) imply that $v_1^{rr}(p-) > \bar{v}_1'(p-)$. As $v_1^{rr}(\tilde{p}_{i-1}) = v_1(\tilde{p}_{i-1}) \geq \bar{v}_1(\tilde{p}_{i-1})$, this implies that there exists a $\hat{p} \in [\tilde{p}_{i-1}, \tilde{p}_i)$ such that $v_1^{rr}(\hat{p}) = \bar{v}_1(\hat{p})$ and $v_1^{rr}(\hat{p}+) < \bar{v}_1'(\hat{p}+)$, a contradiction to (15) and (13). By the same token, suppose that there exists a belief $p \in (\tilde{p}_{i-1}, \tilde{p}_i]$ such that $v_2(p) = \bar{v}_2(p)$. As $s > pg_2$, the relevant ODEs ((15) and (17)) imply that $\bar{v}_2'(p-) > v_2'(p-)$. As $\bar{v}_2(\tilde{p}_{i-1}) \geq v_2(\tilde{p}_{i-1})$, this

implies that there exists a $\hat{p} \in [\tilde{p}_{i-1}, \tilde{p}_i)$ such that $v_2(\hat{p}) = \bar{v}_2(\hat{p})$ and $v_2'(\hat{p}+) > \bar{v}_2'(\hat{p}+)$, a contradiction to (15) and (17).

Now, let $i-1 \geq 2$ be even. Note that our previous step implies that $\bar{v}_2(\tilde{p}_{i-1}) > v_2(\tilde{p}_{i-1})$ and $\bar{v}_1(\tilde{p}_{i-1}) < v_1(\tilde{p}_{i-1})$. Suppose that there exists a $p \in (\tilde{p}_{i-1}, \tilde{p}_i]$ such that $v_n(p) = \bar{v}_n(p)$ for an $n \in \{1, 2\}$. As $(k_1, k_2) = (1, 0)$ on $(\tilde{p}_{i-1}, \tilde{p}_i]$, this immediately implies that $v_n(\tilde{p}_{i-1}) = \bar{v}_n(\tilde{p}_{i-1})$, a contradiction.

On $(\tilde{p}_k, p_S^j]$, a similar argument to the case of even (odd) $i-1$ applies if $j = 2$ ($j = 1$), so that we can conclude that $\bar{v}_1 < v_1$ and $\bar{v}_2 > v_2$ on $(\tilde{p}_1, p_S^j]$, and hence $p_S^1 < p_1'$ and $p_S^2 > p_2'$.

For $l = 1$, from the equilibrium characterisation we know that $p_S^2 = p_2'$ and the above argument to show $p_S^1 < p_1'$ still applies.

B.7 Proof of Proposition 7

If player i ($i = 1, 2$) experiments and player j ($j = 1, 2, j \neq i$) free rides then the players' aggregate equilibrium payoff is given by $v_{agg} = v_i + v_j$, with v_i satisfying the ODE (15) and v_j satisfying the ODE (17). If both players experiment then $v_{agg} = v_1 + v_2$ and v_n ($n = 1, 2$) satisfy the ODE (13).

From Proposition 5, we know that $v_{agg}^c(\tilde{p}_1) = v_{agg}^{nc}(\tilde{p}_1)$. Suppose $v_{agg}^c(\tilde{p}_{i-1}) \geq v_{agg}^{nc}(\tilde{p}_{i-1})$ for some $i \in \{2, 3, \dots, k\}$. Suppose first that $i-1 \geq 1$ is odd. If there exists a $p \in (\tilde{p}_{i-1}, \tilde{p}_i]$ such that $v_{agg}^c(p) = v_{agg}^{nc}(p)$, then by the ODEs (15) and (17), we can conclude that $v_{agg}^{c'}(p-) > v_{agg}^{nc'}(p-)$. This implies there exists a $\hat{p} \in [\tilde{p}_{i-1}, p)$ such that $v_{agg}^c(\hat{p}) = v_{agg}^{nc}(\hat{p})$ and $v_{agg}^{c'}(\hat{p}+) < v_{agg}^{nc'}(\hat{p}+)$, a contradiction to ODEs (15) and (17).

Suppose $i-1 \geq 2$ is even. Then from the previous step we can infer that $v_{agg}^c(\tilde{p}_{i-1}) > v_{agg}^{nc}(\tilde{p}_{i-1})$. In both kinds of equilibria, if $i-1$ is even, $(k_1, k_2) = (1, 0)$ on $(\tilde{p}_{i-1}, \tilde{p}_i]$. This implies that we have $v_{agg}^c(p) > v_{agg}^{nc}(p)$ for all $p \in (\tilde{p}_{i-1}, \tilde{p}_i]$. Thus, for all $p \in (\tilde{p}_1, \tilde{p}_k]$, $v_{agg}^c(p) > v_{agg}^{nc}(p)$.

As $\lambda_2 \leq \lambda_2^*$, we have $\tilde{p}_k = p_S^1$. An argument similar to that for even $i-1$ shows that $v_{agg}^c > v_{agg}^{nc}$ on $p \in (p_S^1, p_2']$. Now, suppose that there exists a $\hat{p} \in (p_2', p_S^2]$ such that $v_{agg}^c(\hat{p}) = v_{agg}^{nc}(\hat{p})$. By the ODEs (15) and (13), this implies $v_{agg}^{c'}(\hat{p}+) > v_{agg}^{nc'}(\hat{p}+)$. This leads to a contradiction by the same argument as above. As $(k_1, k_2) = (1, 1)$ prevails in both equilibria on $(p_S^2, 1)$, the claim follows.

B.8 Proof of Proposition 8

The function w satisfies $w = 2s$ on $[0, \bar{p}_1^*]$, $v = 2s + \bar{B}_1 - \hat{c}$ on $(\bar{p}_1^*, \bar{p}_2^*]$ and $w = 2s + \bar{B}_1 - \bar{c} + \bar{B}_2 - \bar{c}$ on $(\bar{p}_2^*, 1]$; thus, w is the payoff function associated with the policy k^* .¹¹ We shall first show that w is of

¹¹As in Appendix A.1, from 8 we can obtain the ODEs that w satisfies for each range of beliefs and the corresponding general form of w for that range.

class C^1 , (strictly) increasing and (strictly) convex (on $(\bar{p}_1^*, 1)$).

As in Proposition 1, we can compute for $p \in (\bar{p}_1^*, \bar{p}_2^*)$, $\bar{B}_1(p, v) - \bar{c}(p) = \phi(p)$, where ϕ is defined as

$$\phi(p) = -s + p\hat{g} + \frac{1}{r}p \left[2\hat{g} - s - \frac{2\hat{g} + r\hat{g}}{1+r} + \frac{s}{1+r} + \frac{r}{p} \left(s - \bar{p}_1^* \left(\frac{2\hat{g} + r\hat{g}}{1+r} - \frac{s}{r+1} \right) \right) \frac{u_1(p)}{u_1(\bar{p}_1^*)} \right].$$

Direct computation shows that $u_1'' > 0$ and $s - \bar{p}_1^* \left(\frac{2\hat{g} + r\hat{g}}{1+r} - \frac{s}{r+1} \right) > 0$, so that ϕ , and hence $w|_{(\bar{p}_1^*, \bar{p}_2^*)}$ is strictly convex. One furthermore shows by direct computation that $\phi(\bar{p}_1^*) = \phi'(\bar{p}_1^*) = 0$, implying that $w|_{(0, \bar{p}_2^*)}$ is of class C^1 .

We shall now show that \bar{p}_2^* is well-defined. Indeed, by definition, $x = \bar{p}_2^*$ must satisfy

$$s + \left[\frac{2\hat{g} + r\hat{g}}{1+r} - \frac{s}{1+r} \right] x + \left[s - \left(\frac{2\hat{g} + r\hat{g}}{1+r} - \frac{s}{1+r} \right) \bar{p}_1^* \right] \frac{u_1(x)}{u_1(\bar{p}_1^*)} + \frac{\hat{g}[1 - \lambda_2]}{\lambda_2} x = \frac{\lambda}{\lambda_2} s.$$

The left-hand side of this equation is strictly increasing in x for $x > \bar{p}_1^*$ and equal to $s + \frac{s}{\lambda_2} \frac{[\hat{g}(2\lambda_2 + r) - 2\lambda_2 s]}{[\hat{g}(2+r) - 2s]} < \frac{\lambda}{\lambda_2} s$ at $x = \bar{p}_1^*$. Furthermore, at $x = \frac{s}{\hat{g}}$, the left-hand side exceeds $s + \left[\frac{2\hat{g} + r\hat{g}}{1+r} - \frac{s}{1+r} \right] \frac{s}{\hat{g}} > \frac{\lambda}{\lambda_2} s$. By continuity, the equation thus admits of a unique root $\bar{p}_2^* \in (\bar{p}_1^*, \frac{s}{\hat{g}})$.

As $\bar{p}_2^* < \frac{s}{\hat{g}}$, $\frac{\lambda}{\lambda_2} s - \frac{\hat{g}[1 - \lambda_2]}{\lambda_2} \bar{p}_2^* - 2\hat{g}\bar{p}_2^* > 0$, and $w|_{[\bar{p}_2^*, 1]}$ is strictly convex as well. It remains to show that $w|_{[\bar{p}_2^*, 1]}$ is also strictly increasing. By convexity, it is sufficient to show smooth pasting at \bar{p}_2^* . From 8, we can infer that for the region $(\bar{p}_1^*, \bar{p}_2^*)$ (Equation 12 in Appendix A.1), $\bar{p}_2^*(1 - \bar{p}_2^*)w'(\bar{p}_2^* -) = \left[rs + r\bar{p}_2^*\hat{g} + \bar{p}_2^*2\hat{g} - (r + \bar{p}_2^*) \left[\frac{\lambda}{\lambda_2} s - \frac{\hat{g}(1 - \lambda_2)}{\lambda_2} \bar{p}_2^* \right] \right]$. Similarly, for the region $(\bar{p}_2^*, 1)$, we find $\bar{p}_2^*(1 - \bar{p}_2^*)w'(\bar{p}_2^* +) = \left[(r + \lambda)\bar{p}_2^*2\hat{g} - (r + \lambda\bar{p}_2^*) \left[\frac{\lambda}{\lambda_2} s - \frac{\hat{g}(1 - \lambda_2)}{\lambda_2} \bar{p}_2^* \right] \right] / \lambda$, and hence $w'(\bar{p}_2^* +) = w'(\bar{p}_2^* -)$.

It remains to show that w solves the Bellman equation, i.e. that $\bar{B}_i \leq \bar{c}$ for both $i \in \{1, 2\}$ on $[0, \bar{p}_1^*]$; $\bar{B}_1 \geq \bar{c}$ and $\bar{B}_2 \leq \bar{c}$ on $(\bar{p}_1^*, \bar{p}_2^*]$; and $\bar{B}_i \geq \bar{c}$ for both $i \in \{1, 2\}$ on $(\bar{p}_2^*, 1]$. First, consider $p \in [0, \bar{p}_1^*]$. In this case, $w = 2s$, and $\bar{B}_i \leq \bar{c}$ if and only if $p \leq \frac{rs}{r\hat{g} + \lambda_i(2\hat{g} - 2s)}$, which is verified for all $p \leq \bar{p}_1^*$. Now, consider $p \in (\bar{p}_1^*, \bar{p}_2^*]$. As w is strictly increasing in this range, $w = 2s + \bar{B}_1 - \bar{c} > 2s$, and thus $\bar{B}_1 > \bar{c}$. Moreover, $w = 2s + \bar{B}_1 - \bar{c}$ implies that $\bar{B}_2 = \lambda_2 \bar{B}_1 = \lambda_2 (w - s - p\hat{g}) \leq s - p\hat{g} = \bar{c}$ if and only if $w \leq \frac{\lambda}{\lambda_2} s - \frac{\hat{g}p[1 - \lambda_2]}{\lambda_2}$, which is verified as $p \leq \bar{p}_2^*$. Finally, let $p \in (\bar{p}_2^*, 1)$. In this range, we have that $2\hat{g} - w - (1 - p)w' = \frac{r}{\lambda p} \left(\frac{\lambda}{\lambda_2} s - \bar{p}_2^* 2\hat{g} \right) \frac{u_0(p)}{u_0(\bar{p}_2^*)}$, so that $\bar{B}_2 = \frac{\lambda_2}{\lambda} w - \frac{\lambda_2}{\lambda} 2\hat{g}p$, which exceeds $\bar{c} = s - p\hat{g}$ if and only if $w \geq \frac{\lambda}{\lambda_2} s - \frac{\hat{g}p[1 - \lambda_2]}{\lambda_2}$. By monotonicity of w , $w \geq \frac{\lambda}{\lambda_2} s - \frac{\hat{g}p[1 - \lambda_2]}{\lambda_2}$ in this range, which implies $\bar{B}_2 \geq \bar{c}$. Also, $\bar{B}_1 = \frac{\bar{B}_2}{\lambda_2} > \bar{B}_2 \geq \bar{c}$. This completes the proof.

B.9 Proof of Proposition 9

As for Proposition 3, we need to establish that $\bar{p}_2 \geq \bar{p}_1$ if and only if $\lambda_2 > \hat{\lambda}_2$, for some $\hat{\lambda}_2 \in (0, 1)$. As λ_2 increases, \bar{D}_1 (\bar{D}_2) continuously rotates clockwise (anti-clockwise). As \bar{w}_i ($i = 1, 2$) are independent of λ_2 , it follows immediately that \bar{p}_1 (\bar{p}_2) is increasing (decreasing) in λ_2 . As $\lambda_2 \uparrow 1$, \bar{D}_1 and \bar{D}_2 coincide in the limit, and, since $\bar{w}_2 > \bar{w}_1$ on $(\hat{p}_1, 1)$, this implies $\bar{p}_2 < \bar{p}_1$. On the other hand, as $\lambda_2 \downarrow 0$, $\bar{p}'_1 \rightarrow \hat{p}_1$ and $\bar{p}'_2 \rightarrow \frac{s}{\delta} > \hat{p}_1$. This concludes the proof.