Strategic investment and learning with private information*

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Abstract

We study a two-player game of strategic experimentation with private information in which agents choose the timing of risky investments. Agents learn about future returns through privately observed signals, others' investment decisions and from public experimentation outcomes when returns are realized. We characterize symmetric equilibria, and relate the extent of strategic delay of investments in equilibrium to the primitives of the information structure. Agents invest without delay in equilibrium when the most optimistic interim belief exceeds a threshold. Otherwise, delay in investments induces a learning feedback that may either raise or depress beliefs and investment choices. We show that private information in strategic experimentation can increase ex-ante welfare.

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1 Introduction

Learning from peer experience is an important contributor to the proliferation of innovative technology, and as such promotes economic development and growth. That observational learning plays a crucial role in the diffusion of innovation is empirically well-documented. Health professionals learn about medical innovations from the experience of their colleagues (Becker, 1970), households learn about new consumer products from friends and neighbors (Liu et al., 2014; Goolsbee et al., 2002), farmers learn about the qualities of new types of crop from the performance of their peers (Conley and Udry, 2010) and law-makers take into account the experience with legislation in other countries (Aidt and Jensen, 2009).

Theoretical literature has extensively studied the effects of observational learning from peers on individual incentives to engage in costly experimentation (e.g., Bolton and Harris, 1999; Keller and Rady, 2010; Klein and Rady, 2011; Keller and Rady, 2015). In these models, agents continuously decide how much of a valuable resource to invest into a technology with returns drawn from an uncertain distribution. This literature typically presumes symmetry in information, which, from an empirical perspective, is a relevant abstraction from reality. Differences in information, in schooling or personal experience, for example, have been found to be an important determinant in the adoption of new technologies (Foster and Rosenzweig, 2010). In this paper, we propose a tractable model of strategic experimentation with private information. Two decision-makers decide on the timing of an investment into a technology that generates returns drawn from a fixed distribution. The distribution of returns is the same for both agents, but not known to either of them. Each decision maker has access to some initial private information about future returns. After an agent makes the investment, he receives a continuous payoff flow. If the technology is of high quality, he receives a positive return until he stops experimenting with the technology. However, if the technology is of low quality, its use will inevitably lead to a disastrous failure associated with a significant loss for the owner.

The specification of our model is both natural and tractable. Our model captures important features of many real-world settings that involve an opportunity to invest in a new technology (e.g., a drug, chemical, mining procedure) with obvious benefits but unknown, and potentially disastrous, side-effects. The presence of asymmetric information and set-up costs in such a setting gives rise to an initial signaling stage. Due to our bad-news learning specification (Keller and Rady, 2015), agents become more optimistic over time, and experimentation will typically continue after investment unless a failure occurs, in which case the state is revealed and all experimentation stops. Bad-news learning, combined with strategic investment timing in the presence of set-up costs, thus tractably separates the game into an early signaling phase and a later experimentation phase.

We characterize symmetric equilibria in our game. We show that the way private information is aggregated in equilibrium crucially depends on the most optimistic interim belief. If this belief exceeds a certain threshold, optimistic agents invest without delay, so that all private information that is revealed in equilibrium is revealed in a single lump at time zero. If the most optimistic interim belief lies below the threshold, even optimistic agents delay their initial investment and private information is aggregated gradually. The speed at which information is conveyed through signaling varies with time and its evolution depends on the prior belief. For optimistic prior beliefs exceeding a given threshold, learning accelerates and its speed eventually shoots towards infinity, so that all information is conveyed by some finite point in time. Otherwise learning gradually slows down and eventually goes to zero as time approaches infinity.

Inefficiencies arise in our setting because players would rather invest second, leading to under-investment in a classical war of attrition. A key finding in our paper is that asymmetric information can mitigate these inefficiencies. Indeed, with asymmetric information, players on the one hand want to *signal* optimism, so as to encourage investment by their opponent. On the other hand, uncertainty about his opponent's type leads players to keep the *optimistic type* of the opponent indifferent in mixed-strategy equilibrium, whatever his (unknown) true type may be; with public information, by contrast, it is the *actual type* of the opponent that is kept indifferent. The higher investment rates stemming from these combined two effects will, for certain parameters, over-compensate for the losses arising from players' less precise information under private information and thus lead to higher ex-ante welfare.

A relatively small number of papers have investigated the role of private information in games of informational externalities.[1] Rosenberg et al. (2013) investigate a game of strategic experimentation with exponential two-armed bandits, where players' action choices are public information, while the outcomes of these actions are private information. A player's switch to the safe arm is irreversible. They show that, in their setting, public information is unequivocally good for welfare. Their setup differs from ours inter alia by the fact that their players accrue private information over time, while ours are privately informed at the outset. Dong (2017) considers a good-news learning model in the spirit of Keller et al. (2005) in which it is commonly known that one of the two players is privately informed at the outset. Thomas (2019) also analyzes a two-player game with exponential bandits à la Keller et al. (2005), where however there is only one safe arm between the players. In her setting, the quality of the risky arms are independent across players and payoffs are privately observed. Players have an interest in convincing their opponent that their risky arm has already produced a success by delaying stopping. In equilibrium, players mix in a way that tempers the increase in their opponent's belief that they have already achieved a success. Heidhues et al. (2015) investigate the problem with public reversible actions and private payoffs while allowing for cheap-talk communication among players. They

[1]The problem of strategic information acquisition in bandit games has been introduced by Bolton and Harris (1999) in a Brownian-motion environment. Keller et al. (2005) have extended the analysis to a Poisson setting, while Keller and Rady (2015) have introduced "bad-news" Poisson events. Private information in this setting has also been analyzed in Rosenberg et al. (2007).

show that equilibria with public information can always be replicated under private information so that private information is unequivocally good for welfare. This conclusion heavily depends on their assumption that players can communicate with each other. In contrast, Bonatti and Hörner (2011) analyze the case of unobservable and reversible actions and observable outcomes and find that private information boosts welfare in their setting. The reason is that, with observable actions, shirking by a player will render the other players more optimistic, and hence more willing to pick up the slack. This, in turn, makes deviating more attractive under public information. Bonatti and Hörner (2017), by contrast, introduce unobservable actions into Keller and Rady (2015)'s bad-news setting, and find that private information about players' action choices is unequivocally bad for welfare in that setting. Indeed, shirking with observable actions makes the other players less optimistic. Thus, players have stronger incentives to work with public than with private information in a bad-news setting. In our setting, by contrast, we find that private information about the players' initial signals may enhance welfare even in a context of bad-news learning.

In Chamley and Gale (1994) and Murto and Välimäki (2011, 2013), information is dispersed throughout society, and agents only make a single decision, i.e., when to exit the game. Once a player has exited, he is no longer affected by others' decisions. As in our setting, information is inefficiently aggregated because investors have incentives to delay their exit decision so as to acquire more information by observing the behavior of others. The players do not observe the results of their partners' experimentation directly. In our setting, by contrast, players continue to be affected by others' actions after making their investment decision and thus have incentives to influence their partner's actions and beliefs when making that decision.

A closely related paper is Décamps and Mariotti (2004), who study a two-player game of irreversible investments. Private information in their setting, however, pertains to the players' idiosyncratic investment costs, while all information concerning the common quality of the investment opportunity is public. Since, as in our setting, players get additional (public) information after the other player has invested, they prefer the role of the follower and thus have incentives to convince each other that their own costs of investment are high. Once they have made their irreversible investment decision, Décamps and Mariotti (2004)'s players do not care about their partner's actions any longer. In our setting, however, a player prefers his partner to invest as soon as possible, even conditionally on having invested himself. Thus, in contrast to Décamps and Mariotti (2004), our players have incentives to render their partners as optimistic as possible concerning the common investment prospects.

Moscarini and Squintani (2010) consider a model of a winner-takes-all R&D competition in which firms observe an initial private signal about the unknown type of a research project, which is drawn from a continuous distribution. Over time, firms learn about the project's type from their competitor's actions and the lack of success in the past, deciding when to exit irreversibly. They show that the aggregate duration of experimentation is longer under private information, when firms may also exit simultaneously, a non-generic outcome under public information. Since players take no action besides deciding when to exit irreversibly, their action choice is not impacted by signaling motives, in contrast to our setting.

Wagner (2018) and Margaria (2017) analyze related settings, which are also distinguished from our setting by the signaling role of experimentation. Indeed, in Wagner (2018), all learning and experimentation stop after an investment has been made. In Margaria (2017), agents' private signals arrive over time and represent fully conclusive bad news. Therefore, when an agent invests, it will be commonly known that he has not received a bad signal before, and there is nothing more left to learn from him.

The potential welfare improvement of private information is related to the "smoothing effect of uncertainty" (Morris and Shin, 2002). Teoh (1997) demonstrates this effect in a model of public-good provision, where the marginal return to agents' investments is determined by an uncertain state of the world. The author shows that non-disclosure of information may increase ex-ante welfare when the investment has marginally diminishing returns because the loss resulting from a reduction in investment after the release of bad news outweighs the benefits from increased investment when the information is favorable. This is the same mechanism that drives the main result in our paper: when bad news is publicly disclosed, free-riding and leadership-aversion increase, leading to an over-proportional reduction in the expected value of an investment.

2 Model

There are two agents, indexed $i = 1, 2$. Time $t \in \mathbb{R}_+$ is continuous, with an infinite horizon. Future payoffs are discounted at the common discount rate $r > 0$. Each agent decides when to initiate a project which generates a stochastic payoff stream that depends on an unknown state of the world $\theta \in \{G, B\}$, which is either "good" ($\theta = G$) or "bad" ($\theta = B$). Agents can choose to start or end the project at any time, but every time the project is initiated, an investment of size $I \in (0, y/r)$ is required. While the project is operational, it yields a flow return of $y > 0$ in either state. However, when the state is bad, accidents occur at random times corresponding to the jump times of a time-homogeneous Poisson process with parameter $\gamma > 0$. Accidents never occur in state G. Conditionally on the state being B, the arrival times of accidents are independent across agents. An agent whose project causes an accident incurs a lump-sum cost of $c > 0$. The agents' common prior belief that the state is G is $p_0 \in (0, 1)$. At the outset, each agent $i = 1, 2$ receives a signal $s^i \in \{g, b\}$, which provides information about the realization of the state. Either agent's signal is correct (i.e., is equal to g in state G and equal to b in state B) with probability $\rho \in (1/2, 1)$. We assume that conditionally on θ, the signal realizations are independent across agents. Because signal g is positively correlated with the good state G, and b is positively correlated with the bad state B, we call g a "good" signal and b a "bad" signal. Moreover, we commonly call an agent who observed a good signal "optimistic" and an agent who observed a bad signal "pessimistic."

We model the continuous-time environment as a repeated stopping game with multiple "phases". At the beginning of the first phase, the agents decide how long to wait before making the investment, conditionally on the other agent not having invested yet. The initial stage ends after the first agent invests or both invest simultaneously. If only one agent invests, then the agent who invested is called the "leader", and the other the "follower". In the second phase, each agent who invested decides if and when to exit while an agent who did not invest decides when to enter, each conditionally on the other agent not moving first. Later phases proceed in a similar fashion. We assume that $\gamma c > y$, so that after an accident has arrived and players have learned that $\theta = B$, it is a dominant action for players not to invest, or, respectively, to exit a prior investment immediately. We take this as given in our subsequent analysis, and treat all histories following an accident as terminal histories.

Formally, the structure of the game is as follows. We define an investment history at time $t \geq 0$ to be a profile $h_t = ((i_1, \tau_1), \ldots, (i_{n_t}, \tau_{n_t}))$ with $0 \leq \tau_1 \leq \ldots \leq \tau_{n_t} \leq t$, where τ_k for each $k = 1, \ldots, n_t$ represents a "switching time" at which agent $i_k \in \{1, 2\}$ has changed his investment decision, and $n_t \in \mathbb{N}$ represents the total number of instances of such changes in the past. We refer to n_t as the *length* of history h_t. A behavioral strategy for agent i is then given by a family of cumulative distribution functions $\{F_i(\cdot|s_i, h_t)\}_{h_t \in H_t}$ with $F_i(t'|s_i, h_t) = 0$ for all $t' < \tau_{n_t}$. Here, $F_i(t'|s_i, h_t)$ represents the probability that agent i with signal s_i takes action (invests or exits) before or at time $t' \in [\tau_{n_t}, \infty]$ following investment history h_t, conditionally on the other agent $-i$ not taking action before t'. A profile of behavioral strategies induces a distribution over switching times for each agent i. Denoting by $(\tau_k^i)_{k \in \mathbb{N}_0}$ the random investment and exit times for player i, the expected normalized payoff for agent i at any time t is

$$\mathbb{E}_t \left[\sum_{k=0}^{\infty} \left(\int_{\tau_{2k}^i \vee t}^{\tau_{2k+1}^i \vee t} e^{-r(\xi - t)} r(y - \mathbf{1}_{\{\theta = B\}} \gamma c) d\xi - \mathbf{1}_{\{\tau_{2k+1}^i > t\}} e^{-r(\tau_{2k+1}^i - t)} r I \right) \Bigg| s_i, h_t \right]. \tag{1}$$

We say that an agent is "invested" at any history at which he has performed an odd number of switches. Otherwise this agent is called "out".

Our solution concept is symmetric perfect Bayesian equilibrium. A *perfect Bayesian equilibrium* is a pair of behavioral strategies, together with a system of beliefs for each agent, which assigns a probability distribution over signals and the state of the world at each history, such that (i) each agent's strategy maximizes his expected payoff, given his belief over the state and the other agent's signal and (ii) beliefs are updated via Bayes' rule at any history that lies in the support of the distribution over histories induced by the agents' strategies. We shall say that a perfect Bayesian equilibrium is *symmetric* if the players' equilibrium strategies prescribe the same (mixed) action whenever they have the same beliefs and are in the same *mode*, that is, they are either both invested or both out.

Throughout, we denote by p_t the (history-dependent) public posterior belief that $\theta = G$ at time t, i.e., the belief held by a hypothetical outside observer, who started out with a prior belief of p_0 and observed the public history but did not know about the initial signals. By the same token, we denote by q_{it} the public posterior belief assigned to agent i's type being g (we omit the index i whenever the belief is the same for each agent). Furthermore, we write $p_t(s)$ and $q_{it}(s)$ for the respective posterior probabilities conditional on a single signal $s \in \{g, b\}$, and, analogously, $p_t(s, s')$ for the posterior probability about the state, conditional on a pair of signals $(s, s') \in \{g, b\}^2$. Note that, since signals are i.i.d. and symmetric, we have $p_t = p_t(g, b) = p_t(b, g)$.

3 Equilibrium analysis

3.1 Public information

We first consider the case in which both signals are publicly observable. This scenario will serve as a benchmark for the case with privately observed signals, and it allows us to establish connections to existing models of strategic experimentation without private information. When signals are publicly observed, the agents share a common belief about the state after observing the realization of signals. We denote their common belief by $\check{p}_0 := p_0(s^1, s^2)$. Consider an investment history at some time t at which $k \geq 1$ agents are currently invested. Since investments and the arrival of accidents are publicly observable, both agents update their belief about the realization of the state of the world based on the observed actions and payoffs. In the absence of any accidents following an investment, the posterior belief $\check{p}_t$ continuously evolves following the familiar differential equation

$$\frac{d\check{p}_t}{dt} = k\gamma\, \check{p}_t(1 - \check{p}_t).$$

Throughout, we assume that the investment I that is required to initiate a project is large enough to ensure that agents remain invested after they initiated their project (unless an accident occurs), even if the other agent deviates from his equilibrium strategy. We begin by constructing a symmetric equilibrium with this property, showing that for sufficiently large I, this equilibrium is indeed unique in the class of symmetric equilibria.

We can derive a symmetric equilibrium without exits by backwards induction. At an investment history at which both agents are already invested, both remain invested indefinitely. At a history at which only one agent invested, the leader remains invested indefinitely, and the follower decides how long to wait before making the investment. Before the first investment, each agent must decide how long to wait, conditionally on the other agent not having invested first. To construct our equilibrium, we thus need to derive the follower's optimal investment delay. We then proceed to find a distribution over initial investment times that are mutually optimal given the continuation play, verifying that it is indeed never optimal for either agent to exit prior to an accident.

We begin by considering a history at which one agent is invested ($k = 1$). The follower benefits from the leader's experimentation in this case, because of the possibility that the leader experiences an accident. If the leader experiences an accident, the follower learns that the state of the world is bad without incurring any losses. Depending on the follower's posterior belief about the state, it may thus be profitable for him to delay the investment. Assuming that the follower delays his own investment by some duration τ (at which he is sufficiently confident that the state is good), the expected net present value before paying investment costs I for the leader at any belief p is given by

$$v_l(p,\tau) = py + (1-p)\left((1-e^{-(r+\gamma)\tau})\lambda_1 + e^{-(r+\gamma)\tau}\lambda_2\right)(y-\gamma c) - rI. \quad (2)$$

where by $\lambda_k = r/(r+k\gamma)$ we denote the marginal value of a discounted unit payoff stream up to termination at a random time arriving at constant rate $k\gamma$ for $k = 1, 2$. By the same token, assuming that the leader remains invested indefinitely, the expected present value of the follower when delaying the investment by a duration τ is given by

$$v_f(p,\tau) = e^{-r\tau}py + e^{-(r+\gamma)\tau}(1-p)\lambda_2(y-c\gamma) - (p+(1-p)e^{-\gamma\tau})e^{-r\tau}rI. \quad (3)$$

Define the log-likelihood ratio $\phi(p) := \ln(p) - \ln(1-p)$. The following lemma reports basic properties of the functions v_l and v_f.

Lemma 1. *The function $v_l(p,\tau)$ is linearly increasing in p, convex and decreasing in τ for every $p \in (0,1)$ and supermodular in (p,τ). The function $v_f(p,\tau)$ is linearly increasing in p and has a single peak in τ at*

$$\tau^*(p) = \begin{cases} \left(\phi(p_f^*) - \phi(p)\right)/\gamma & \text{if } p < p_f^* \\ 0 & \text{if } p \geq p_f^* \end{cases} \quad (4)$$

for every $p \in (0,1)$, where

$$p_f^* = \frac{\lambda_1(rI+\gamma I) + \lambda_2(\gamma c - y)}{\lambda_1(y+\gamma I) + \lambda_2(\gamma c - y)}. \quad (5)$$

All proofs are found in the Appendix. We write $v_f^*(p) = v_f(p,\tau^*(p))$ and $v_l^*(p) = v_l(p,\tau^*(p))$ for the values of the leader and the follower, respectively, given the follower uses the optimal delay. Since τ^* is weakly decreasing in p, and v_l and v_f are strictly increasing in p as well as decreasing in τ, it follows that v_l^* and v_f^* are strictly increasing functions in p. Moreover, v_l^* is continuous, positive if $p = 1$, and negative if $p = 0$. Hence, it has a unique root on $(0,1)$, which we denote by p_l^*. Note that, by definition of p_f^* and p_l^*, we have $v_f^*(p_f^*) = v_l^*(p_f^*) > 0$ and thus $p_l^* < p_f^*$.

Consider now a history of the game, in which neither agent has invested, so that $k = 0$. In this case, each agent anticipates in this conjectured equilibrium that, once they invest, the follower will delay their investment by $\tau(\check{p}_0)$. If $\check{p}_0 < p_l^*$, then neither agent is willing to invest and become the leader in any equilibrium in which players invest at most once, by definition of p_l^*. If $p_l^* < \check{p}_0 < p_f^*$, then the value of becoming the leader is positive. Note, however, that there can be no symmetric equilibrium in pure strategies in which players invest at most once, as the best response to the other agent's investment would be not to invest and vice-versa. The same argument rules out atoms in mixed-strategy equilibrium. Thus, if there is a symmetric equilibrium, it must be in atomless mixed strategies, with each agent investing at a rate that renders the other agent indifferent between investing immediately and delaying his investment by any length of time. As long as neither agent has made the investment, no new information becomes available, so that the agents' equilibrium flow rate of investment β is constant over time. We can immediately calculate the equilibrium investment rate, using the fact that each agent must be indifferent between making the investment immediately and delaying investment by another instant. This implies that each agent must invest at a rate β that solves

$$v_l^*(\check{p}_0) = \beta v_f^*(\check{p}_0)dt + (1 - rdt - \beta dt)v_l^*(\check{p}_0).$$

Solving the equation for β gives the investment rate as a function of the common belief $\check{p}_0$:

$$\beta^*(\check{p}_0) = \max\left\{\frac{rv_l^*(\check{p}_0)}{v_f^*(\check{p}_0) - v_l^*(\check{p}_0)}, 0\right\}. \tag{6}$$

It follows from $\check{p}_0 < p_f^*$ and Lemma 1 that the denominator of β^* is always strictly positive, and, therefore, the rate of investment is positive whenever the value of becoming the leader is greater than zero. Moreover, the difference between the follower's and the leader's value converges to 0 as $\check{p}_0$ approaches the threshold p_f^*, so that the investment rate $\beta^*(\cdot)$ approaches infinity as $\check{p}_0 \to p_f^*$. Note also that the payoff for each agent in this case is, by construction, equal to $v_l^*(\check{p}_0)$.

To verify that this is indeed an equilibrium, we need to check that no agent wants to exit once he has invested. Both agents are invested whenever $\check{p}_0 \geq p_f^*$; by definition of p_f^*, no agent can gain from exiting. When $\check{p}_0 < p_f^*$ and only one agent has invested, then the leader would receive at most $v_f^*(\check{p}_t)$ after exiting, while he obtains $v_l^*(\check{p}_t) + rI$ if he remains indefinitely if no accident occurs. If I is close to y/r, however, then $v_f^*(\check{p}_t)$ is close to zero. On the other hand, $v_l^*(\check{p}_t) + rI$ is close to y, since $v_l^*(\check{p}_t) \geq 0$, for otherwise an agent would not want to become leader in the first place. Thus, for I close to y/r, we have $v_l^*(\check{p}_t) + rI > v_f^*(\check{p}_t)$, which implies that the leader cannot gain from exiting.

The above equilibrium turns out to be unique provided the investment I required to initiate the project is sufficiently large. A large investment deters agents from making frequent switches, and thus makes it costly for the agents to respond to deviations by their opponent. When I is small, it is possible to construct an equilibrium in which both agents invest immediately, and every deviation is punished with immediate exit by the competitor. For large values of I, exiting in response to a deviation by the opponent is not a credible threat and thus cannot be part of any equilibrium.

A necessary condition for simultaneous investment to be part of an equilibrium is that the payoff for each agent be non-negative. An agent's payoff from jointly investing immediately, given a posterior belief $\check{p}_t$, is given by

$$v_l(\check{p}_t, 0) = \check{p}_t y + (1 - \check{p}_t)\lambda_2(y - \gamma c) - rI. \tag{7}$$

This payoff is non-negative if and only if $\check{p}_t \geq \underline{p}$, where

$$\underline{p} = \frac{rI + \lambda_2(\gamma c - y)}{y + \lambda_2(\gamma c - y)}. \tag{8}$$

Clearly, there can be no initial investment in equilibrium when the prior belief $\check{p}_0$ lies below $\underline{p}$, since then the payoff from investing is necessarily negative for each agent.

Now, define $\hat{p}_l^*$ to be the lowest posterior belief at which the payoff of already being invested as the leader is non-negative, i.e. $v_l^*(\hat{p}_l^*) + rI = 0$. The following result shows that, for I sufficiently large, the thresholds $\hat{p}_l^*$, $\underline{p}$, p_f^* and p_l^* defined above satisfy the following chain of inequalities.

Lemma 2. *There is an $I_0 \in (0, y/r)$ such that, for $I \geq I_0$, we have*

$$\hat{p}_l^* < \underline{p} < p_l^* < p_f^* < 1.$$

The lemma implies in particular that, for I sufficiently large, joint investment can never arise in equilibrium at any posterior belief below the threshold p_f^*, because each agent correctly anticipates that the opponent would never again exit following the investment (except after a failure). If agent 1, for example, was to invest at some posterior $\check{p}_t < p_f^*$, then agent 2 would prefer to wait and become the follower, knowing that agent 1 would not want to exit.

The following theorem summarizes these findings, and characterizes the unique symmetric equilibrium for symmetrically informed agents.

Theorem 1 (Symmetric equilibrium with public information)**.** *There exists $I^* \in (0, y/r)$ such that, for all $I > I^*$, there is a symmetric equilibrium, in which neither agent exits before the arrival of an accident. If $\check{p}_0 \geq p_f^*$, both agents invest immediately. If $p_l^* < \check{p}_0 < p_f^*$, each agent invests at constant rate $\beta^*(\check{p}_0)$ given by Equation (6) in the first phase, while the follower starts the project with delay $\tau^*(\check{p}_0)$. If $\check{p}_0 \leq p_l^*$, neither agent invests. This equilibrium is unique in the class of symmetric equilibria.*

The equilibrium structure mirrors that of the symmetric equilibrium in Keller and Rady (2015), in the sense that there are two belief thresholds with the property that there is no experimentation below the first, and maximum experimentation above the second threshold, with randomization for all beliefs that lie between these thresholds. For intermediate prior beliefs, the first phase of the game is strategically similar to standard "war-of-attrition" games (see, e.g., Bulow and Klemperer, 1999).

Note that we focus our analysis on the case in which the investment cost I is large. This assumption is motivated by both practical relevance as well as technical considerations. The strategic experimentation literature tends to limit attention to stationary Markov perfect equilibria, in which the agents' strategies are time-invariant functions of their posterior belief about the state of the world. Aside from making the analysis more tractable, the purpose of focussing on Markov perfect equilibria is to isolate effects that result from information spill-overs.[2] In our model, however, the players' strategies in any Markov perfect equilibrium must condition on the opponent's current binary mode of experimentation, that is, whether or not the other player is currently invested. But this means that players can punish deviations directly, so that, by focussing on Markov perfect equilibrium, one no longer effectively isolates effects from information spill-overs. For example, if there are no investment costs, there exist Markov perfect equilibria in which both players invest immediately and are deterred from exiting by the threat of immediate exit by the opponent. Such punishment becomes non-credible when players must make large investments to initiate experimentation. Once a player begins experimentation by investing, the investment cost is sunk, and he subsequently no longer finds it worthwhile to stop and re-pay this cost in order to punish his opponent for deviating.

3.2 Private information

We now turn to equilibria in the case of private signals. The equilibria differ from the case of publicly observed signals in that the presence of private information exacerbates uncertainty and introduces signaling incentives. Uncertainty is enhanced because the agents are both less informed about the state of the world. Signaling incentives arise due to learning spill-overs and social learning: since each agent benefits from the other's experimentation, each has an incentive to behave in a way that makes the other agent more optimistic in order to encourage him to engage in more experimentation.

[2] An exception is Hörner et al. (2018), who analyze non-Markovian equilibria in this context.

In the equilibria we characterize, the way private information is revealed depends crucially on whether the most optimistic interim belief $p_0(g, g)$ exceeds the follower threshold p_f^* or not. If $p_0(g, g) \geq p_f^*$, optimistic agents invest without delay, so that all private information that is revealed in equilibrium is revealed in a lump at time zero. Otherwise, optimistic agents delay their initial investment while pessimistic agents wait, so that private information is aggregated continuously over time.

The following result characterizes three different classes of equilibria and conditions for their existence that depend on the prior belief $p_0 \in (0, 1)$ and the signal precision $\rho \in (1/2, 1)$. Note that, depending on the parameters, the rate $\beta^*(p_0(b, b))$ may be 0.

Theorem 2 (Symmetric equilibrium with private information)**.** *There is a $I^{**} \in (0, y/r)$ such that, for all $I > I^{**}$, there exists an equilibrium in which no agent exits prior to an accident, followers delay by τ^* and the following holds.*

(1.) *Suppose $p_0 \geq p_l^*$ and $\rho \in (1/2, 1)$ are such that $p_0(g, g) \geq p_f^*$. Then there exists a symmetric equilibrium in which type g invests immediately. Type b of each agent invests immediately with some probability $\eta^* \in [0, 1]$, and with probability $1 - \eta^*$, he invests at a random time arriving at constant rate $\beta^*(p_0(b, b))$.*

(2.) *Suppose $p_0 < p_l^*$ and $\rho \in (1/2, 1)$ are such that $p_0(g, g) \geq p_f^*$. Then there exists a symmetric equilibrium in which type g invests immediately with some probability $\nu^* \in [0, 1]$, and with probability $1 - \nu^*$ does not invest. Type b never invests.*

(3.) *Suppose $p_0 > 0$ and $\rho \in (1/2, 1)$ are such that $p_0(g, g) < p_f^*$. Then there exists a symmetric equilibrium in which type g of each agent invests at rate $\mu_t^* \geq 0$ given by (21). Type b delays investment until (possibly infinite) $t^* > 0$, given by (22), and invests at constant rate $\beta^*(p_0(b, b))$ thereafter.*

We refer to the first two types of equilibria as equilibria with "immediate investment," and to the latter as an equilibrium with "delayed investment."

Whether the equilibrium exhibits delayed investment depends the relative location of $p_0(g,g)$ vs. p_f^*.

Equilibria with immediate investment arise either when signals are very informative or if the prior belief is very high. Such equilibria may be pooling, partially separating, or fully separating. A high prior belief and weak signals result in pooling, where each agent invests immediately. Indeed, if the interim belief of pessimistic agents, conditional on their own signal, is high enough, then it is always optimal for them to invest immediately. On the other hand, when signals are highly informative, the equilibrium tends to be partially or fully separating. Intuitively, an informative good signal provides a strong incentive for an agent to invest, while an informative bad signal makes investing costly. However, immediate investment communicates good news that makes one's opponent more willing to invest, which, in turn, generates a positive informational externality. As a result, pessimistic agents have incentives to pretend to be optimistic, in order to encourage the other agent to experiment.

Equilibria exhibit delayed investment when the prior is not too high and signals not too informative. In an equilibrium with delayed investment, optimistic agents engage in an attrition game, while pessimistic agents simply wait; indeed, as in the case of public information, optimistic agents delay investment because they benefit from the possibility that their opponent invests first and then subsequently provides free information. In contrast to the case of public information, however, in equilibrium, an optimistic agent invests at a rate that keeps his opponent's *optimistic type* indifferent, irrespectively of his opponent's true type, while, with public information, agents keep each other's true types indifferent. Moreover, because only optimistic agents invest with positive probability, the agents learn about one another's types while they wait. As long as neither agent invests, each agent becomes gradually more certain that the other one has observed a bad signal. This gradual change in beliefs about the other's type, in turn, affects the agents' incentives to invest. The interaction between belief updating, incentives and actions creates a learning feedback loop that either accelerates or dampens the speed of learning.

The effects of the feedback loop can be seen in the dynamics of invest-

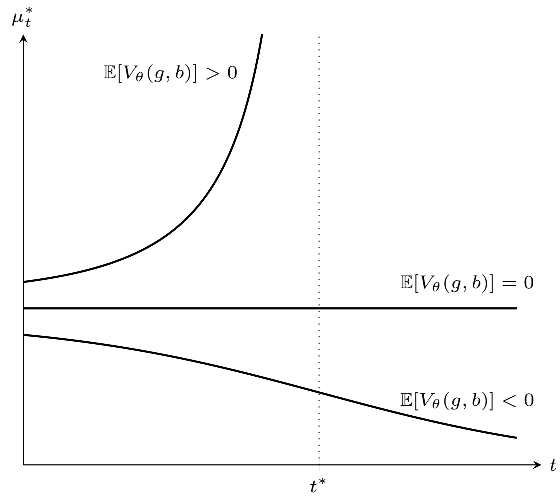

Figure 1: Three branches of equilibrium investment rates.

ment rates illustrated in Figure 1. Here $\mathbb{E}[V_\theta(s_i, s_{-i})]$ denotes the equilibrium payoff from being a leader in state θ for an agent with signal s_i, when his opponent's signal is s_{-i}. The upper branch in the figure corresponds to the case in which the expected equilibrium payoff $\mathbb{E}[V_\theta(g,b)] > 0$ from being an optimistic leader, conditional on the opponent's signal being bad, is positive. In this case, an agent of type g wants to invest regardless of his opponent's type. In equilibrium, each optimistic agent invests at a rate that makes his optimistic opponent just indifferent between investing and waiting. The longer an optimistic agent waits for the other to invest first, the more convinced he becomes that the other's delay is due to his signal being bad. In equilibrium, therefore, optimistic agents must increase their rates of investment in order to continue to make the good type of the other agent indifferent between waiting and investing. This increase, in turn, accelerates the decline in beliefs, which requires a further increase in the investment rate of optimistic agents. The result is an escalating feedback-loop between investment and learning rates, which causes investment rates to shoot off to infinity, so that all private information is revealed by some finite time t^*.

The lower branch in Figure 1 corresponds to the case in which type g's worst-case continuation payoff $\mathbb{E}[V_\theta(g,b)]$ from an immediate investment is negative. As before, the longer an optimistic agent waits for the other to invest first, the more convinced he becomes that the other's deferral is due to his signal being bad. In this case, the expected value of becoming the leader diminishes over time. In order for optimistic agents to continue to be indifferent between waiting and investing, they must decrease their rate of investment gradually. This reduction again triggers a feed-back loop, in which decreasing investment rates slow down learning, which in turn dampens investments and so on. Investment rates eventually tend to 0 and the agents' private information is never fully revealed.

If $\mathbb{E}[V_\theta(g,b)] = 0$, finally, a type-$g$ agent would be indifferent between investing and staying out if he knew his partner to be of type b. In this case, agents of type g invest at a constant rate in equilibrium, so that agents' beliefs that their partner is of type g decline over time, yet all private information is only revealed in the limit as $t \to \infty$.

4 Welfare and transparency

In this section, we consider welfare properties of the equilibria derived in the previous sections. Our notion of efficiency corresponds to a setting in which agents pool their private information and commit to a strategy at the outset of the game, seeking to maximize the sum of their payoffs. The following result characterizes this cooperative solution.

Theorem 3 (Cooperative benchmark)**.** *There exist thresholds $p_1^* < p_l^*$ and $p_2^* \in (p_1^*, p_f^*)$, so that it is socially optimal for both agents to invest immediately if $\check{p}_0 \geq p_2^*$ and never to invest if $\check{p}_0 < p_1^*$. If $p_1^* \leq \check{p}_0 < p_2^*$, then it is socially optimal for one agent to invest immediately, and for the second agent to invest with delay $\tau^s(\check{p}_0) = (\phi(p_2^*) - \phi(\check{p}_0))/\gamma$.*

It is noteworthy here that, due to switching costs, staggered investment is optimal for intermediate values of the interim beliefs. Since the initial investment costs required to start a project cannot be recovered after a failure, it is socially preferable to start only one project initially, which then generates a flow of information on which the start of the second project can be conditioned. In this way, staggered investment lowers the loss from making irreversible investments in the bad state. The cooperative solution shares the feature of staggered investment with the equilibrium under public information. However, in equilibrium, there is a period of inefficient delay prior to the initial investment. Moreover, the equilibrium exhibits too little experimentation relative to the efficient benchmark. On the one hand, since $p_1^* < p_l^*$, there are values of the interim belief at which experimentation is socially valuable but does not arise in equilibrium. Second, since $p_2^* < p_f^*$, delay of the second investment is inefficiently long. Inefficiencies arise in equilibrium due to free-riding incentives: agents benefit from the information generated by their competitor's experimentation, and they fail fully to internalize the social value of their own experimentation. The incentive to free-ride thus leads to inefficiently long delays in investment by the follower, and results in sluggish initial investment, as each agent prefers the other to invest first.

In comparing the equilibrium outcomes with and without information asymmetries, it is natural to ask which environment is more desirable from an efficiency standpoint. Naïve logic may suggest that more transparency should unambiguously lead to better outcomes, as it allows the agents to make better-informed decisions. This view, however, disregards the aforementioned positive side-effects of private information. The following result formalizes this insight. It shows that private information is socially preferable if investment costs are substantial, the prior belief that the state is good is high enough, and the signals are not too informative. Indeed, denote by $W_0(s^1, s^2)$ the expected social surplus generated in the unique symmetric equilibrium under public information, when $p_0(s_1, s_2)$ is the common initial belief that the state is good. Let $\tilde{W}$ denote the ex-ante expected social surplus generated in a symmetric equilibrium in which each agent's signal is private information. We then have the following

Theorem 4 (Welfare improvement through private information)**.** *Fix $p_0 > p_l^*$. There exists $\rho^* > 1/2$ and $\bar{I} \in (0, \frac{y}{r})$ such that, for all signal precisions*

$\rho \in (1/2, \rho^*)$ and investment costs $I \in (\bar{I}, \frac{y}{r})$, we have $\tilde{W} \geq \mathbb{E}[W_0(s_1, s_2)]$. For $p_l^* < p_0 < p_f^*$, the inequality is strict.

The condition on the signal precision ensures that it is socially optimal for both agents to invest (though it may not necessarily be socially optimal for both to invest right away). If $p_0 \geq p_f^*$, there always exists a range of signal precisions guaranteeing that both types of agents invest immediately under private information, as in the efficient benchmark. If $p_l^* < p_0 < p_f^*$, then there are signal precisions such that the symmetric equilibrium under public information has delayed entry. In this case, a mixed-strategy equilibrium arises both under public and under private information. In this equilibrium players invest at a rate that keeps the actual type of their opponent indifferent, if information is public. If information is private, by contrast, type g of player i invests at a rate that would keep type g of player $-i$ indifferent, even if player $-i$ happens to be of type b. The proof establishes that the welfare benefit of this increased investment rate of the g-types will overwhelm the welfare loss both from the reduced investment rate of b-types and from players' less precise knowledge of the underlying state.

It is well known that, in the presence of payoff externalities, more information can lead to socially inferior equilibrium outcomes, as uncertainty will typically relax incentive-compatibility constraints. By focussing on the case of high I, however, we are able to isolate the effects of informational spill-overs; thus, here, the welfare gain from private information ensues from a mechanism that is purely informational in nature.

5 Conclusion

We propose a tractable model of strategic experimentation with private information and bad-news learning in the presence of non-negligible switching costs. We derive the unique symmetric equilibrium in the case of symmetric information. We also construct equilibria for the case of privately observed signals, which exhibit either immediate or randomly delayed investment. We trace these properties back to an encouragement-through-signaling effect and to strategic uncertainty resulting from asymmetric information. Finally, we show that, due to these effects, equilibrium surplus can be higher with private information.

There are a number of natural extensions we do not address in this paper. For example, we do not allow communication between agents. However, we conjecture that communication would not change the equilibrium in our model, since agents would always send the most positive signal to induce more experimentation by their partner, so that, in equilibrium, all communication would amount to babbling. We also do not address any questions regarding games with more than two agents. For public information, we conjecture that an equilibrium with many agents would be characterized by an increasing sequence of belief thresholds, where each indicates the belief at which the next investment takes place with certainty. Equilibria with private information will also be affected by encouragement and strategic uncertainty in this case, and we expect that many of the aspects described in this paper will carry over to a game with more than two players. However, the welfare implications are ambiguous, because the information aggregation problem becomes more severe, while the social value of experimentation increases.

6 Proofs

Proof of Lemma 1. (i) That v_l is linear in p is obvious from its definition in Equation (2) and v_l is increasing in p because it follows from $\gamma c > y > 0$ that the second term in Equation (2) is negative. To see that v_l is decreasing in τ, note that $\lambda_2 < \lambda_1$, and hence

$$\frac{d}{d\tau}v_l(p,\tau) = -(r+\gamma)(1-p)(\lambda_1-\lambda_2)e^{-(r+\gamma)\tau}(\gamma c - y) < 0$$

for all $p \in (0,1)$ and $\tau \geq 0$. That v_l is convex in τ for all $p \in (0,1)$ follows from

$$\frac{d^2}{d^2\tau}v_l(p,\tau) = (r+\gamma)^2(1-p)(\lambda_1-\lambda_2)e^{-(r+\gamma)\tau}(\gamma c - y) > 0.$$

Finally, supermodularity holds because

$$\frac{d^2}{dpd\tau}v_l(p,\tau) = (r+\gamma)(\lambda_1-\lambda_2)e^{-(r+\gamma)\tau}(\gamma c - y) > 0.$$

(ii) Linearity of v_f in p is obvious from its definition in (3) and it is increasing in p because $\gamma c > y > 0$ implies that the first term in Equation (3) is positive and the second term is negative. For fixed $p \in (0,1)$, the derivative of v_f with respect to τ is

$$\frac{d}{d\tau}v_f(p,\tau) = -e^{-r\tau}\left[rpy-(r+\gamma)e^{-\gamma\tau}(1-p)\lambda_2(c\gamma-y)\right]+e^{-r\tau}rI[rp+(r+\gamma)(1-p)e^{-\gamma\tau}].$$

Let $\hat{\tau}(p)$ be the (finite) solution to the first order condition $dv_f(p,\tau)/d\tau = 0$. The second term in brackets is positive, so that $dv_f(p,\tau)/d\tau > 0$ if $\tau < \hat{\tau}(p)$ and $dv_f(p,\tau)/d\tau < 0$ if $\tau > \hat{\tau}(p)$. Hence, v_f attains a global maximum at $\hat{\tau}(p)$. If $\hat{\tau}(p) \geq 0$, then solving

$$rp(y-rI)+(r+\gamma)e^{-\gamma\hat{\tau}^*(p_0)}(1-p)(\lambda_2(y-c\gamma)-rI) = 0$$

for $\hat{\tau}(p)$ shows that $\hat{\tau}(p) = \tau^*(p)$. If $\hat{\tau}(p) < 0$, then $v_f(p,\cdot)$ is strictly decreasing on $[0,\infty)$, and therefore assumes its maximum at 0. □

Proof of Lemma 2. Note that for all $p < p_f^*$ we have $v_l^*(p) < v_l(p,0) = v_f(p,0) < v_f^*(p)$ with equality everywhere when $p = p_f^*$. By definition, we have $v_l(\underline{p},0) = 0$, $v_l^*(p_l^*) = 0$. Since v_l, v_f, v_l^*, and v_f^* are all continuously increasing functions, the first inequality $v_l^*(p) < v_l(p,0)$ implies that $\underline{p} < p_l^*$. By definition of p_f^*, we have $v_f^*(p_f^*) > 0$, and thus the identity $v_l^*(p_f^*) = v_f^*(p_f^*)$ implies that $p_l^* < p_f^*$. Finally, when $I \to y/r$ then $\underline{p} \to 1$, while $\hat{p}_l^* < 1$ is bounded away from 1. Hence, there is an $I_0 < y/r$, such that $\hat{p}_l^* < \underline{p}$. □

Lemma 3. *In every equilibrium with public signals, each agent invests immediately at any belief* $\check{p}_0 \geq p_f^*$.

Proof. It is clear that for $\check{p}_0 = 1$, it is a unique best-response for each agent to invest immediately. Because of switching costs, there also exists a threshold

$p_0^\dagger \in (p_f^*, 1)$ close to one, such that an agent who is invested at a belief will not exit at any $\check{p}_0 \geq p_0^\dagger$. At a history with posterior belief $\check{p}_0 \geq p_0^\dagger$ at which exactly one agent is invested, the agent who is out will thus invest immediately. At a history with posterior belief $\check{p}_0 \geq p_0^\dagger$ at which both agents are out, they anticipate that the other agent will invest immediately following their own investment, and thus each strictly prefers to invest immediately. Thus, in any equilibrium, both agents are invested at any belief $\check{p}_0 \geq p_0^\dagger$. Because of switching costs, there exists an $\epsilon \in (0, p_0^\dagger - p_f^*)$ such that any history with posterior belief $\check{p}_0 \geq p_1^\dagger := p_0^\dagger - \epsilon$ at which exactly one agent is invested, this agent would not exit. Again, by definition of p_f^*, the agent who is out would thus invest immediately. Again, if both agents are out, they would then invest immediately. Thus, both agents would invest immediately at any belief $\check{p}_0 \geq p_1^\dagger$. The same argument applies for any threshold above p_f^* so that, in any equilibrium, both agents invest immediately at any $\check{p}_0 \geq p_f^*$. □

Proof of Theorem 1. (1.) *Existence*: (i) Let $\check{p}_0 \geq p_f^*$. The claim immediately follows from the definition of p_f^*.

(ii) Let $p_l^* \leq \check{p}_0 < p_f^*$. If an agent who is invested exits, he receives the payoff $v_l^*(\check{p}_t)$ by construction. By the same argument as in Part (2.) below, it is never optimal for a leader to exit when $\check{p}_0 > p_l^*$. Before either agent has invested, we have

$$v_f^*(\check{p}_0) > v_l^*(\check{p}_0) > 0,$$

which implies that each agent strictly prefers being the follower over being the leader, and each prefers being the leader over an outcome in which neither agent ever invests. By symmetry, each agent has to choose the same distribution over switching times. By standard arguments, the equilibrium distribution cannot have any atoms or gaps in its support. Thus, the investment rate $\beta^*(\check{p}_0)$ from Equation (6) characterizes the distribution that makes each agent indifferent between investing and not investing, which establishes the claim.

(iii) For $\check{p}_0 \leq p_l^*$, the claim follows immediately from the definition of p_l^*.

(2.) *Uniqueness*: Consider a history with posterior belief $\check{p}_t \in [\hat{p}_l^*, p_f^*)$, at which exactly one agent, say agent 1, is invested. By Lemma 3, in any equilibrium, agent 2 invests immediately at any $s \geq t$ at which $\check{p}_s \geq p_f^*$. Thus, if agent 1 stays invested indefinitely, (or until an accident occurs,) the largest delay compatible with equilibrium is $\tau^*(\check{p}_t)$. The payoff for agent 1 is therefore no less than $v_l^*(\check{p}_s) + rI$ at each $s \geq t$. Let $(I^k)_{k\in\mathbb{N}_0}$ be an increasing sequence in $[0, y/r]$ with $I^0 = 0$ and $I^k \to y/r$ for $k \to \infty$, and let $(\check{p}_0^k)$ be a sequence of prior beliefs in (p_l^{*k}, p_f^{*k}), where p_l^{*k} and p_f^{*k} are the leader and follower thresholds for investment cost I^k for each k. Further, let $\underline{p}^k$ be given by (8) with $I = I^k$. Since $\underline{p}^k \to 1$ for $k \to \infty$, we have $p_l^{*k} \to 1$ for $k \to \infty$. Thus, the payoff for the leader is at least $v_l^*(\check{p}_0^k) + rI^k \to y$. On the other hand, the best thing that can happen for agent 1 after exiting is that the other agent invests and never exits. Thus, the highest payoff agent 1 can achieve after an exit is $v_f^*(\check{p}_0^k)$ which converges to zero as $I^k \to y/r$. Together it follows that

$$\lim_{k\to\infty} v_l^*(\check{p}_0^k) + rI^k - v_f^*(\check{p}_0^k) = y \tag{9}$$

which implies that there exists a $k^\dagger$, such that for all $k > k^\dagger$, we have

$$v_l^*(\check{p}_0^k) + rI^k > v_f^*(\check{p}_0^k).$$

This inequality implies that a leader cannot gain by exiting at beliefs in the range $[\hat{p}_l^*, p_f^*)$ if $I > I^\dagger$, for some $I^\dagger > 0$. From (9), it follows that $I^\dagger < y/r$. Given that the leader does not exit, there cannot be any $\check{p}_t \in [\hat{p}_l^*, p_f^*)$ at which both agents invest immediately since either agent would prefer delaying the investment and become a follower. Since there is no equilibrium with simultaneous investment, in symmetric equilibrium, both agents must choose the same distribution over initial investment times. Because the continuation strategies are unique, there is a unique investment rate, given by (21), that has the property that each agent is willing to randomize. Thus, there is a unique symmetric equilibrium outcome. By Lemma 2, we have $\hat{p}_l^* < \underline{p}$, for I sufficiently large. In this case, agents' expected payoffs from investing are thus negative for $\check{p}_0 \leq \hat{p}_l^*$. Therefore, in every equilibrium, both agents refrain from investing/ rescind their respective investment for good in this range. □

Proof of Theorem 2. *Part* (1.): Suppose $p_0 > 0$ and $\rho > 1/2$ such that $p_0(g,g) \geq p_f^*$. We proceed to verify that the following strategies and beliefs are part of an equilibrium. In the first phase, an agent with signal g invests immediately. An agent with signal b invests immediately with probability $\eta \in [0,1]$. With probability $1-\eta$, he invests at a random time drawn from an exponential distribution with parameter $\beta^*(p_0(b,b))$. In the second phase of the game, a follower with posterior belief p delays investment by $\tau^*(p)$, and the beliefs at t are updated via Bayes' rule whenever possible. A leader with signal g reverses his investment immediately at $t=0$ (and stays out) if and only if his posterior belief about θ is lower than $\hat{p}_l^*$. Otherwise he stays invested indefinitely. A leader with signal b reverses his investment immediately at $t=0$ with some probability $1-\nu \in [0,1]$. Either agent who remains invested in the second (or third) phase exits after the occurrence of a failure. Unless otherwise stated, beliefs after off-path histories are specified as follows: In any phase, at time t, the non-deviating agent with signal s assigns probability $p_t(b,s)$ to state $\theta = H$. By the same token, after any off-path exit of agent i, agent $-i$ assigns probability 1 to agent i's signal being b. Any off-equilibrium investment of agent i does not affect agent $-i$ belief about agent i's signal. The deviating agent's beliefs do not change as a result of his deviation.

We show that there exist $\eta, \nu \in (0,1)$ such that the above strategies and associated beliefs characterize a perfect Bayesian equilibrium. Consider first the second phase, taking as given that each type g invests at $t=0$ with probability 1, and each type b invests at $t=0$ with probability η and waits with probability $1-\eta$. As shown in Lemma 1, the function $\tau^*(p)$ is the optimal delay of the follower with posterior p, and thus given the leader stays in the game, a follower cannot gain from deviating in the second phase of the game. If the leader exits immediately in the second phase, the follower cannot influence that decision.

Suppose agent i with signal s invests at time $t=0$ in the first phase, and the other agent $-i$ does not, so that at $t=0$ in the second phase, agent i is the leader and $-i$ the follower. If both expect that the other agent follows the strategy described in the previous paragraph, then the posterior belief of agent i with signal s is $p_0(b,s)$, and the posterior belief of type s of agent $-i$ is, by Bayes' rule,

$$\tilde{p}(\eta, s) = \frac{p_0(s)(\rho + \eta(1-\rho))}{p_0(s)(\rho + \eta(1-\rho)) + (1-p_0(s))(1-(1-\eta)\rho)}. \qquad (10)$$

If $v_l(p_0(b,b), \tau^*(\tilde{p}(\eta,b))) + rI > v_l^*(p_0(b,b)) + rI > 0$, then the continuation payoffs for either type of agent i is positive (since $v_l(p_0, \tau^*(\tilde{p}(\eta\nu, b))) > v_l(p_0(b,b), \tau^*(\tilde{p}(\eta\nu, b)))$, and both types of agent i remain invested for sure. If, on the other hand, $v_l(p_0(b,b), \tau^*(\tilde{p}(\eta,b))) + rI < 0 < v_l(p_0(b,b), \tau^*(p_0)) + rI$, then type b of agent i remains in the game with probability $\nu^* \in (0,1)$ solving $v_l(p_0(b,b), \tau^*(\tilde{p}(\eta\nu^*, b))) + rI = 0$.[3] If $v_l(p_0(b,b), \tau^*(p_0)) + rI < 0$, then type b of agent i exits for sure, and type g remains for sure.

We need to show that there exists a value for η such that neither agent can gain by deviating from the specified strategies in the first phase. Denote by $V_\theta(\eta)$ the value of investing at $t=0$ for an agent in state θ, and let $W_\theta(\eta,\tau)$ be the value of waiting at $t=0$ when the agent delays investment by τ as follower.Note that here τ refers to the delay of the investment, given that the other agent invests immediately at $t=0$. When neither agent invests, then each agent is convinced that the other agent's type is bad, so that there is no longer any uncertainty about the other's private information, and the unique symmetric equilibrium under public information with $\check{p}_0 = p_0(b,b)$ is played after that history. Note here that when the state and strategies are given, the payoff is independent of private signals. Note also that for a follower in the second phase, the optimal delay for type b is $\tau^*(\tilde{p}(\eta,b))$ when type b of the other agent invests with probability η. We write

$$\mathbb{E}[W_\theta^*(\eta)|s] := \max_\tau \mathbb{E}[W_\theta(\eta,\tau)|s] = \mathbb{E}[W_\theta(\eta, \tau^*(\tilde{p}(\eta,s))|s].$$

for the payoff from waiting when using the optimal delay.

There is a pooling equilibrium, i.e. $\eta = 1$, if and only if $p_0(b) \geq p_f^*$, since,

[3]Note that $\tilde{p}(\eta\nu^*, b)$ is the posterior belief of type b of agent $-i$ that the state is H, conditional on the joint event that agent i invested in the first phase and remains in the second phase, where $\eta\nu^*$ is the probability that type b of agent i does this.

in this case, each type of each agent is willing to invest immediately, if the other agent invests for sure, and thus reveals no information. Thus, consider the case $p_0(b) < p_f^*$. Note that in this case, we have $\mathbb{E}[W_\theta^*(1)|b] \geq \mathbb{E}[V_\theta(1)|b]$, since bad types always have incentives to wait when the other agent invests with probability one. There are two cases to consider, $\mathbb{E}[W_\theta^*(0)|b] < \mathbb{E}[V_\theta(0)|b]$ and $\mathbb{E}[W_\theta^*(0)|b] \geq \mathbb{E}[V_\theta(0)|b]$.

(i.) Suppose $\mathbb{E}[W_\theta^*(0)|b] < \mathbb{E}[V_\theta(0)|b]$, i.e., an agent with a bad signal prefers to invest immediately in the first phase at time zero, if the other agent invests with zero probability after a bad signal and invests immediately after a good signal. In this case, there exists a partial (or full) pooling equilibrium in which type g always invests while type b invests with probability $\eta^* \in (0, 1]$. Note that the functions $\mathbb{E}[W_\theta^*(\eta)|b]$ and $\mathbb{E}[V_\theta(\eta)|b]$ are convex combinations of continuous functions and hence continuous. Thus, there exists an $\eta^* \in (0, 1]$ such that $\mathbb{E}[V_\theta(\eta^*) - W_\theta^*(\eta^*)|b] = 0$, so that an agent with type b is indifferent between investing and not investing, given the other agent invests with probability η^* after observing signal b. We shall now verify an agent of type g's incentives to invest. Note that we have the following inequality:

$$
\begin{aligned}
0 &= \mathbb{E}[V_\theta(\eta^*) - W_\theta^*(\eta^*)|b] \leq \mathbb{E}[V_\theta(\eta^*) - W_\theta(\eta^*, \tau^*(\tilde{p}(\eta^*, g)))|b] \\
&= p_0(b)\big(V_H(\eta^*) - W_H(\eta^*, \tau^*(\tilde{p}(\eta^*, g)))\big) + (1 - p_0(b))\big(V_L(\eta^*) - W_L(\eta^*, \tau^*(\tilde{p}(\eta^*, g)))\big) \\
&\leq p_0(g)\big(V_H(\eta^*) - W_H(\eta^*, \tau^*(\tilde{p}(\eta^*, g)))\big) + (1 - p_0(g))\big(V_L(\eta^*) - W_L(\eta^*, \tau^*(\tilde{p}(\eta^*, g)))\big) \\
&= \mathbb{E}[V_\theta(\eta^*) - W_\theta^*(\eta^*)|g],
\end{aligned}
$$

where the first inequality follows from the fact that $\mathbb{E}[W_\theta^*(\eta^*)|b] \geq \mathbb{E}[W_\theta(\eta^*, \tau)|b]$ for all $\tau \geq 0$ by definition, and the second inequality from $p_0(g) > p_0(b)$, and from the fact that investing immediately is strictly better than waiting if and only if the state is H (since there is no gain from delay in state H, and no gain from investing in state L).

(ii.) Now, suppose $\mathbb{E}[W_\theta^*(0)|b] \geq \mathbb{E}[V_\theta(0)|b]$, so that agents with signal b prefer to wait if the other agent invests only if his signal is g. For agents with signal g in this case we have

$$\mathbb{E}[V_\theta(0) - W_\theta(0)|g] = q_0(g)\big(v_l(p_0(g,g),0) - v_f^*(p_0(g,g))\big) \\ + (1-q_0(g))\big(0 \vee v_l^*(p_0) - 0 \vee v_l(p_0, \tau^*(p_0(b,b))\big).$$

Since $p_0(g,g) \geq p_f^*$ by assumption, we have $v_l(p_0(g,g),0) - v_f^*(p_0(g,g)) = 0$, and thus $\mathbb{E}[V_\theta(0) - W_\theta(0)|g] \geq 0$. Thus, in this case, we have a fully separating equilibrium in which g-types invest at $t=0$, whereas b-types do not.

It remains to be shown that, in both cases (i.) and (ii.), if agent i has incentives to invest at time $t=0$, then he has no incentive subsequently to exit, provided I is large enough. Similarly to the proof of Theorem 1, let $(I^k)_{k\in\mathbb{N}_0}$ be an increasing sequence in $[0, y/r]$ with $I^0 = 0$ and $I^k \to y/r$ for $k \to \infty$, and let (p_0^k, ρ^k) be a sequence of information structures with $p_l^{*k} < p_0^k < p_f^{*k} < p_0^k(g,g)$, where p_l^{*k} and p_f^{*k} are, respectively, the leader and follower thresholds for investment costs I^k, such that either $\mathbb{E}[W_\theta^{*k}(0)|b] < \mathbb{E}[V_\theta^k(\eta)|b]$ or $\mathbb{E}[W_\theta^{*k}(0)|b] \geq \mathbb{E}[V_\theta^k(\eta)|b]$ for all $k \geq 0$, where $W_\theta^{*k}(\eta), V_\theta^k(\eta)$ denote the follower and leader value for each k (as above). Moreover, let $\tilde{p}^k(\eta, b)$ be the posterior belief given by (10) at step k, and let $\eta^k \in [0,1]$ be the critical value with the property that (1) for $p_0(b) < p_f^{*k}$, either type b of each agent is indifferent or else $\eta^k = 0$, and (2) for $p_0(b) \geq p_f^{*k}$, we have $\eta^k = 1$. Finally, let $\underline{p}^k$ be given by (8) with $I = I^k$. Note that, if $s_i = b$, our assumption that i has incentives to invest at time $t = 0$ implies $\tilde{p}^k(\eta^k, b) \geq \underline{p}^k$ for all k. If $s_i = g$, our assumption in the statement of the theorem implies that i's belief after any history is bounded below by $p_0 \geq p_l^{*k} \geq \underline{p}^k$.

(i.) *Both agents invested.* Note that $\tilde{p}^k(\eta^k, b) \to 1$ as $\underline{p}^k \to 1$, for $k \to \infty$. Thus, the payoff of remaining invested, for either type of agent, is at least $v_l^*(\tilde{p}^k(\eta^k, b), 0) + rI^k \to y$. On the other hand, the best thing that could happen to an agent after exiting would be for the other agent to reveal his type, to invest and never to exit. Thus, the highest payoff either type of agent i could possibly achieve after an exit is $\mathbb{E}[v_f^*(p_0^k(s_{-i}, g)]$ which converges to zero as $I^k \to y/r$. Together it follows that

$$\lim_{k\to\infty} v_l^*(\tilde{p}^k(\eta^k, b), 0) + rI^k - \mathbb{E}[v_f^*(p_0^k(s_{-i}, g)] = y \tag{11}$$

which implies that there exists a $\tilde{k}_1$, such that for all $k > \tilde{k}_1$, we have

$$v_l^*(\tilde{p}^k(\eta^k, b), 0) + rI^k > \mathbb{E}[v_f^*(p_0^k(s_{-i}, g)].$$

This inequality implies that there exists a threshold $\tilde{I}_1$ such that a leader of either type cannot gain by exiting if $I > \tilde{I}_1$. From (11), it follows that $\tilde{I}_1 < y/r$.

(ii.) *Only one agent invested.* First, we argue that type b of each agent i remains invested if he invests himself and his payoff as leader is positive, i.e., if $p_0^k(b, b) > \hat{p}_l^{*k}$. If $p_0^k(b, b) \in [\hat{p}_l^{*k}, p_l^{*k})$, then exit is clearly not optimal, since after i's exit, agent $-i$ (whose type has become known to be b after he did not invest) will never invest going forward. Thus assume $p_0^k(b, b) \geq p_l^{*k}$. Then $v_l^*(p_0^k(b, b)) + rI^k \geq v_l^*(p_l^{*k}) + rI^k$. Since $\underline{p}^k \to 1$ for $k \to \infty$, it follows from $p_0^k(b, b) \geq \underline{p}^k$ that $p_0^k(b, b) \to 1$ for $k \to \infty$, and therefore, $v_l^*(p_0^k(b, b)) + rI^k \to y$. On the other hand, the best thing that could happen for agent i after exiting is that the other agent invests and never exits. Thus, the highest payoff agent i can achieve after an exit is at most $v_f^*(p_t^k(b, b))$ which converges to zero as $I^k \to y/r$. Together it follows that

$$\lim_{k\to\infty} v_l^*(p_t^k(b, b)) + rI^k - v_f^*(\tilde{p}^k(\eta^k, b)) = y \tag{12}$$

which implies that there exists a $\tilde{k}_0$, such that for all $k > \tilde{k}_0$, we have

$$v_l^*(p_0^k(b, b)) + rI^k > v_f^*(\tilde{p}^k(\eta^k, b)).$$

This inequality implies that there exists a threshold $\tilde{I}_0$ such that a leader cannot gain by exiting if $I > \tilde{I}_0$. From (12), it follows that $\tilde{I}_0 < y/r$.

Part (2.): Consider symmetric strategies with the following properties. Each agent with signal g invests with probability ν, and each agent with signal b waits indefinitely in the first phase. If $p_0 \geq \hat{p}_l^*$, an agent who invested at $t = 0$ always remains invested until an accident occurs; if $p_0 < \hat{p}_l^*$, an agent i who invested at $t = 0$ in the first phase rescinds his investment at $t = 0$ in the third phase if and only if $-i$ did not invest in the second phase at $t = 0$. If agent i invests immediately, then agent $-i$ invests without delay if his signal is g. If agent $-i$'s signal is b, then he delays his investment by $\tau^*(p_0)$. Agents never invests after any other history. Each agent with signal g is indifferent between investing and delaying his investment if

$$q_0(g)v_l(p_0(g,g),0)+(1-q_0(g))(\max\{v_l^*(p_0)+rI,0\}-rI) = \nu q_0(g)v_f(p_0(g,g),0).$$

which is equivalent to

$$\nu^* = 1 + \left(\frac{1-q_0(g)}{q_0(g)}\right)\left(\frac{\max\{v_l^*(p_0)+rI,0\}-rI}{v_f(p_0(g,g),0)}\right) \tag{13}$$

When agent i with signal g invests, his continuation strategy is optimal by construction. To show that it is optimal for agents with signal b to wait, denote by $V_\theta(\eta)$ the value of investing at $t = 0$ for an agent in state θ, and let $W_\theta(\eta, \tau)$ be the value of waiting at $t = 0$ when the agent delays investment by τ as follower. By construction of ν^* and τ^*, we have

$$\begin{aligned}
0 &= \mathbb{E}[V_\theta(\nu^*) - W_\theta^*(\nu^*)|g] \\
&= p_0(g)\big(V_H(\nu^*) - W_H(\nu^*, \tau^*(p_0(g,g)))\big) + (1-p_0(g))\big(V_L(\nu^*) - W_L(\nu^*, \tau^*(p_0(g,g)))\big) \\
&\geq p_0(b)\big(V_H(\nu^*) - W_H(\nu^*, \tau^*(p_0(g,g)))\big) + (1-p_0(b))\big(V_L(\nu^*) - W_L(\nu^*, \tau^*(p_0(g,g)))\big) \\
&= \mathbb{E}[V_\theta(\nu^*) - W_\theta(\nu^*, \tau^*(p_0(g,g)))|b] \\
&\geq \mathbb{E}[V_\theta(\nu^*) - W_\theta^*(\nu^*)|b].
\end{aligned}$$

Thus, for an agent with signal b, it is a best response to wait. A similar argument to before establishes that it is never optimal for an agent who invested to exit.

Part (3.): Suppose $p_0 > 0$ and $\rho \in (1/2, 1)$ are such that $p_0(g, g) < p_f^*$. The strategies outlined in the theorem imply that an agent who invests in the first phase and becomes the leader reveals himself to be of type g. Suppose that after agent i invests at time t in the first phase, the agents use the following continuation strategy:

- Agent $-i$ with signal g invests at time $t + \Delta$, where $\Delta = \tau^*(p_0(g, g))$.
- If $v_l^*(p_{t+\Delta}) + rI \geq 0$, where

$$p_{t+\Delta} = \frac{p_0}{p_0 + (1 - p_0)e^{-\gamma\Delta}},$$

  then type g of agent i remains in the game indefinitely, and each type s of agent $-i$ invests with delay $\tau^*(p_0(s, g))$

- If $v_l^*(p_{t+\Delta}) + rI < 0 < v_l(p_{t+\Delta}, 0)$, then type g of agent i remains for sure until $t + \Delta$ and type g of the follower enters after delay Δ. Beginning at time $t + \Delta$, type b of agent $-i$ invests at rate $\phi_f(s)$ solving

$$0 = y - (1 - p_s)\gamma c + \phi_f(s)(v_l(p_s, 0) + rI)$$

  and agent i exits at rate $\phi_l(s)$ solving

$$v_l(p_s, 0) = (1 - rdt - (1 - p)\gamma dt - \phi_l(s)dt)v_l(p_{s+dt}, 0).$$

  The exit and investment rates ϕ_l, ϕ_f are defined in a way that the leader and follower are willing to randomize. Note that since delay is profitable for the follower for all $p < p_f^*$, we have

$$v_l(p_s, 0) < (1 - rdt - \gamma dt)v_l(p_{s+dt}, 0).$$

  and thus $\phi_l > 0$.

- If $v_l(p_{t+\Delta}, 0) + rI, v_l(p_{t+\Delta}, 0) < 0$, then type g of agent i remains in until $t + \Delta$ if agent $-i$ invests with delay Δ, and agent i exits otherwise. Type g of agent $-i$ invests with delay Δ, and type b of agent $-i$ never invests.

If agent i with signal g, who invested at some time $t < t^*$, deviates by exiting, we assume that agent $-i$ delays investment indefinitely while both agents are out, yet ignores the deviation completely as soon as agent i re-enters, making it a best response for the deviating agent to re-invest immediately upon exiting (since it was optimal for him to enter in the first place). Given agent i re-invests immediately, it is a best response for agent $-i$ to stay out.[4]

(i) *Derivation of the equilibrium investment rate of agents with signal g.* Henceforth, for each of the three cases above, denote by $V_\theta(s_i, s_{-i})$ the value of agent i conditional on (1) state θ and (2) agent i with signal s_i being the leader, and agent $-i$ with signal s_{-i} using the assigned follower strategy. Similarly, let $W_\theta(s_i, s_{-i})$ be the value of becoming the follower. Note that these payoffs are independent of time in the first phase, since the signal pair (s_i, s_{-i}) encapsulates all information that is exchanged in the first phase, so that $p_t(s_i, s_{-i}) = p_0(s_i, s_{-i})$. Thus, each agent's expected value of becoming the leader is given by

$$U(q_t(g)) = q_t(g)\mathbb{E}[V_\theta(g,g)|s_i = g, s_{-i} = g] + (1-q_t(g))\mathbb{E}[V_\theta(g,b))|s_i = g, s_{-i} = b].$$

Type g of each agent is willing to randomize if he is indifferent between investing immediately and waiting for another instant. Hence, the value function for type g of the agent must satisfy the indifference condition

$$U(q_t(g)) = \mu_t q_t(g)\mathbb{E}[W_\theta(g,g)|s_i = g, s_{-i} = g]dt + (1 - rdt - \mu_t q_t(g)dt)U(q_{t+dt}(g)). \tag{14}$$

By Ito's Lemma, the indifference condition (14) can be written as

$$U(q_{t+dt}(g)) = U(q_t(g)) + dU(q_t(g))dq_t(g), \tag{15}$$

where by definition of U, we have $dU(q_t(g))/dq_t(g) = \mathbb{E}[V_\theta(g,g)] - \mathbb{E}[V_\theta(g,b)]$.

[4] Note that, since the follower and the leader have divergent beliefs at a history with a single investment, our definition of symmetry imposes no restrictions after such histories.

Bayes' rule implies that the posterior belief at $t + dt$ is

$$q_{t+dt}(g) = \frac{q_t(g)(1 - \mu_t dt)}{1 - q_t(g)\mu_t dt}.$$

The differential change in belief is therefore

$$\frac{dq_t(g)}{dt} \equiv \lim_{dt \to 0} \frac{q_{t+dt}(g) - q_t(g)}{dt} = -\mu_t q_t(g)(1 - q_t(g)). \tag{16}$$

If we now substitute equations (15) and (16) in the indifference condition (14) and ignore higher order terms, we obtain the expression

$$rU(q_t(g)) = \mu_t q_t(g)\mathbb{E}[W_\theta(g,g) - V_\theta(g,g)|s_i = g, s_{-i} = g]. \tag{17}$$

Since $p_0(g,g) < p_f^*$, the right-hand side of this equation is strictly positive. Simplifying and solving the equation for μ_t yields

$$\mu_t = \frac{rU(q_t(g))}{q_t(g)\mathbb{E}[W_\theta(g,g) - V_\theta(g,g)|s_i = g, s_{-i} = g]}. \tag{18}$$

Here, μ_t is the rate of investment for type g of each agent in the symmetric equilibrium at a given belief $q_t(g)$. Note that since $p_0(g,g) < p_f^*$, we have $W_\theta(g,g) > V_\theta(g,g)$, and thus $\mu_t \in [0,\infty)$. (If $p_0(g,g) \geq p_f^*$, then $W_\theta(g,g) \leq V_\theta(g,g)$, and an equilibrium of the type constructed here does not exist.) Substituting this last expression into Equation (16), we obtain the evolution of the posterior $q_t(g)$ in equilibrium:

$$dq_t(g) = -(1 - q_t(g))\frac{rU(q_t(g))}{\mathbb{E}[W_\theta(g,g) - V_\theta(g,g)|s_i = g, s_{-i} = g]}dt. \tag{19}$$

We obtain the equilibrium belief and equilibrium investment rate at each time t by solving Equation (19) with given initial belief q_0. The initial value problem (19) has the unique solution

$$q_t(g) = \frac{e^{-t\beta^*(p_0(g,g))}U(q_0(g)) + (1 - q_0(g))\mathbb{E}[V_\theta(g,b)]}{e^{-t\beta^*(p_0(g,g))}U(q_0(g)) - (1 - q_0(g))\mathbb{E}\left[V_\theta(g,g) - V_\theta(g,b)|s_i = g\right]}. \tag{20}$$

We now substitute $q_t(g)$ into Equation (18) and simplify to obtain the equilibrium rate of investment

$$\mu_t^* = \frac{e^{-t\beta^*(p_0(g,g))}U(q_0(g))}{e^{-t\beta^*(p_0(g,g))}U(q_0(g)) + (1-q_0)\mathbb{E}[V_\theta(g,b)|s_i=g]}\beta^*(p_0(g,g)). \tag{21}$$

If $r\mathbb{E}[V_\theta(g,b)] > 0$ then the investment rate μ_t^* diverges to $+\infty$ as $t \to t^*$, where

$$t^* = \log\left(1 + \frac{p_0(g,g)}{p_0}\frac{\mathbb{E}[V_\theta(g,g)]}{\mathbb{E}[V_\theta(g,b)]}\right)^{\beta^*(p_0(g,g))}. \tag{22}$$

If, on the other hand, $\mathbb{E}[V_\theta(g,b)] < 0$, then μ_t^* converges to 0 as $t \to \infty$. Thus, $t^* = \infty$.

(ii) *Agents with signal b prefer to wait until t^*.* For agents with signal b, the incremental opportunity cost from waiting is $r\mathbb{E}[V_\theta(b,s)|s_i = b]dt$. The expected incremental gain from waiting for this type is $\mu_t^* q_t(b)\mathbb{E}[W_\theta(b,g) - V_\theta(b,g)|s_i = b]dt$. We show that when agents with signal g invest at rate μ_t^*, then agents with signal b prefer to wait:

$$r\mathbb{E}[V_\theta(b,s)|s_i = b] \leq \mu_t^* q_t(b)\mathbb{E}[W_\theta(b,g) - V_\theta(b,g)|s_i = b, s_{-i} = g]. \tag{23}$$

Because flow values are positive in state H and negative in state L, i.e., $y \geq 0 \geq y - \gamma c$, we have $V_H(s_i, s_{-i}) \geq 0 \geq V_L(s_i, s_{-i})$. Therefore:

$$\begin{aligned}
\mathbb{E}_t[V_\theta(s_i,s_{-i})|s_i = b] &= p_t(b)\mathbb{E}_t[V_H(s_i,s_{-i})|s_i = b] + (1-p_t(b))\mathbb{E}_t[V_L(s_i,s_{-i})|s_i = b] \\
&\leq p_t(g)\mathbb{E}_t[V_H(s_i,s_{-i})|s_i = b] + (1-p_t(g))\mathbb{E}_t[V_L(s_i,s_{-i})|s_i = b] \\
&\leq p_t(g)\mathbb{E}_t[V_H(b,s_{-i})|s_i = g] + (1-p_t(g))\mathbb{E}_t[V_L(b,s_{-i})|s_i = g] \\
&\leq p_t(g)\mathbb{E}_t[V_H(g,s_{-i})|s_i = g] + (1-p_t(g))\mathbb{E}_t[V_L(g,s_{-i})|s_i = g] \\
&= \mathbb{E}_t[V_\theta(s_i,s_{-i})|s_i = g].
\end{aligned}$$

The first inequality follows because $p_t(g) > p_t(b)$. Note that according to our prescribed strategies, agents with signal g invest earlier than agents with signal b, and exit later. Therefore, we have $V_\theta(s_i, g) \geq V_\theta(s_i, b)$. Since $q_t(g) > q_t(b)$, we thus have $\mathbb{E}_t[V_\theta(b, s_{-i})|s_i = g] \geq \mathbb{E}_t[V_\theta(b, s_{-i})|s_i = b]$ for each θ, which explains the second inequality. The last inequality follows because the strategy of type g is constructed to maximize the continuation payoff after investing. Note that

$$\frac{1}{q_t(s_i)}\mathbb{E}[V_\theta(s_i, s_{-i})|s_i] = \frac{p_t(s_i)}{q_t(s_i)}\mathbb{E}[V_H(s_i, s_{-i})|s_i] + \frac{1-p_t(s_i)}{q_t(s_i)}\mathbb{E}[V_L(s_i, s_{-i})|s_i]$$

Because $\rho > 1/2$ and $p_t(g) > p_t(b)$, it follows that

$$\frac{p_t(b)}{q_t(b)} = \frac{p_t(b)}{\rho p_t(b) + (1-\rho)(1-p_t(b))} < \frac{p_t(g)}{\rho p_t(g) + (1-\rho)(1-p_t(g))} = \frac{p_t(g)}{q_t(g)}$$

and similarly,

$$\frac{1-p_t(b)}{q_t(b)} > \frac{1-p_t(g)}{q_t(g)}$$

Combining the previous results, we find that

$$\begin{aligned}
\frac{1}{q_t(b)}\mathbb{E}[V_\theta(s_i, s_{-i})|s_i = b] &= \frac{p_t(b)}{q_t(b)}\mathbb{E}[V_H(s_i, s_{-i})|s_i = b] + \frac{1-p_t(b)}{q_t(b)}\mathbb{E}[V_L(s_i, s_{-i})|s_i = b] \\
&\leq \frac{p_t(g)}{q_t(g)}\mathbb{E}[V_H(s_i, s_{-i})|s_i = b] + \frac{1-p_t(g)}{q_t(g)}\mathbb{E}[V_L(s_i, s_{-i})|s_i = b] \\
&\leq \frac{p_t(g)}{q_t(g)}\mathbb{E}[V_H(s_i, s_{-i})|s_i = g] + \frac{1-p_t(g)}{q_t(g)}\mathbb{E}[V_L(s_i, s_{-i})|s_i = g] \\
&= \frac{1}{q_t(g)}\mathbb{E}[V_\theta(s_i, s_{-i})|s_i = g]
\end{aligned}$$

The previous inequalities, together with (17), imply

$$\begin{aligned}
\frac{1}{q_t(b)}r\mathbb{E}[V_\theta(s_i, s_{-i})|s_i = b] &\leq \frac{1}{q_t(g)}r\mathbb{E}[V_\theta(s_i, s_{-i})|s_i = g] && (24) \\
&= \mu_t^*\mathbb{E}[W_\theta(g,g) - V_\theta(g,g)|s_i = g, s_{-i} = g]. && (25)
\end{aligned}$$

(iii) *No exit before failure.* If neither agent has invested at time $t > t^*$, then it is common knowledge that both agents have observed bad signals, and thus there is a unique symmetric continuation equilibrium in this case. Suppose agent i invests at $t < t^*$ and subsequently deviates by exiting. As before, we assume that, after such a history, the non-deviating agent stays out forever while both agents are out, and ignores the deviation as soon as the deviator re-enters, making it a best response for the deviating agent immediately to re-invest. This in turn makes it a best response for the first agent to stay out. □

Proof of Theorem 3. It is obvious that, prior to an accident, it can never be optimal for either agent to switch more than once. The team's objective can therefore be written as

$$w(p, \tau) = \max\{0, v_l(p, \tau) + v_f(p, \tau)\},$$

where τ denotes the delay with which the follower invests. Using Equations (2) and (3), we can write explicitly:

$$w(p,\tau) = (1+e^{-r\tau})py + (1-p)\left[\lambda_1 + (2\lambda_2 - \lambda_1)e^{-(r+\gamma)\tau}\right](y - \gamma c) - (1 + (p + (1-p)e^{-(r+\gamma)\tau}))rI.$$

It follows from the definitions of λ_1 and λ_2 that $2\lambda_2 - \lambda_1 = \lambda_1\lambda_2$. Therefore, the marginal value of delaying the second investment is

$$\frac{\partial w(p,\tau)}{\partial \tau} = re^{-r\tau}[-p(y - rI) + (1-p)(\lambda_2(\gamma c - y) + (r+\gamma)I)e^{-\gamma\tau}]. \tag{26}$$

The expression in brackets is strictly decreasing in τ. This implies that $\partial w(p,\tau)/\partial\tau < 0$ for all $\tau \geq 0$ whenever $p > p_2^*$, where

$$p_2^* = \frac{rI + \gamma I + \lambda_2(\gamma c - y)}{y + \gamma I + \lambda_2(\gamma c - y)}, \tag{27}$$

in which case it is socially optimal to make the second investment immediately. If $p \leq p_2^*$, then the socially optimal delay solves the first-order condition $dw(p,\tau)/d\tau = 0$. Thus, the optimal delay is given by

$$\tau^s(p) = \begin{cases} (\phi(p_2^*) - \phi(p))/\gamma & \text{if } p < p_1^*, \\ 0 & \text{if } p \geq p_1^*. \end{cases}$$

Define $w^s(p) = w(p, \tau^s(p))$. By Lemma 1, the functions v_l and v_f are increasing in p which implies that w^s is an increasing function. Since w^s is continuous with $w^s(0) < 0$ and $w^s(1) > 0$, it follows that there exists a unique threshold p_1^* that solves $w^s(p_1^*) = 0$. $\square$

Proof of Theorem 4. Let P be the distribution over signals for given parameters p_0 and ρ. We write $P(s_i)$ for the probability that a given agent's signal is s_i and $P(s_1, s_2)$ for the probability that the pair of signals is (s_1, s_2).

(1.) Suppose $p_0 > p_f^*$. Then, choose $\rho^* > 1/2$ such that $p_0(b) > p_f^*$ and $p_0(b, b) > p_2^*$. As we show in the proof of Theorem 2, it follows that for all $\rho < \rho^*$, there exists a pooling equilibrium in which each type of each agent invests immediately. By Theorem 3, this equilibrium is efficient. Hence $\tilde{W} \geq E[W_\theta(s_1, s_2)]$.

(2.) Let $p_f^* > p_0 > p_l^*$. Choose $\rho^* > 1/2$ such that $p_f^* > p_0(g, g)$ and $p_0(b, b) \geq p_l^*$. By Theorem 2, there exists an equilibrium with delayed entry. It follows from arguments in the proof of Theorem 2 that the expected equilibrium value of the good type of each agent is $\mathbb{E}[V_\theta(g, s_{-i})] \geq \mathbb{E}[v_l^*|g]$. The inequality follows from the fact that leaders have the option to exit. By Equation (23), bad types strictly prefer to delay investment at each $t < t^*$. The expected payoff for an agent of type b who deviates by investing before time t^* is bounded below by

$$q_t(b)v_l(p_0, \tau^*(p_0(g, g))) + (1 - q_t(b))v_l^*(p_0(b, b), \tau^*(p_0)) > \mathbb{E}[v_l^*|b].$$

Therefore, the expected social surplus for each agent is

$$\tilde{W} > P(g)\mathbb{E}[v_l^*|g] + P(b)\mathbb{E}[v_l^*|b] = \mathbb{E}[W_\theta(s^1, s^2)].$$

(3.) Let $p_0 = p_f^*$. Since $p_0(g, g) > p_f^*$, each agent with signal g invests immediately. We show that there exists $\rho^* > 1/2$ such that $\tilde{W} > E[W_\theta(s_1, s_2)]$ for all $\rho \in (1/2, \rho^*)$. The equilibrium is with immediate investment, since

$p_0(g,g) > p_f^*$. There cannot be a pooling eqilibrium, since $p_0(b) < p_0 = p_f^*$.

The social welfare per agent therefore satisfies the inequality

$$\tilde{W} \geq P(g)\Big[q_0(g)v_l(p_0(g,g),0)+(1-q_0(g))(\eta v_l(p_0,0)+(1-\eta)v_l(p_0,\tau^*(\tilde{p}_\eta(b)))\Big]$$
$$+P(b)\Big[q_0(b)v_l(p_0,0)+(1-q_0(b))(\eta v_l(p_0(b,b),0)+(1-\eta)(v_l(p_0(b,b),\tau^*(p_\eta(b)))\Big].$$

The right-hand side represents the ex-ante expected payoff for an agent who invests immediately after each signal, which is a lower bound for the equilibrium payoff. Note that $P(g)q_0(g) = P(g,g)$, $P(g)(1-q_0(g)) = P(b)q_0(b) = P(b,g)$ and $P(b)(1-q_0(b)) = P(b,b)$. Using $q_0(b)p_0 + (1-q_0(b))p_0(b,b) = p_0(b)$ together with the linearity of $v_l(p,0)$, we can write

$$\begin{aligned}\tilde{W} &\geq P(g,g)v_l(p_0(g,g),0) + P(b,g)v_l(p_0,0)\\ &\quad + (P(g,b)+P(b,b))\Big[\eta v_l(p_0(b),0) + (1-\eta)v_l(p_0(b),\tau^*(\tilde{p}(\eta,b))\Big].\end{aligned}$$

When signals are public, then, after each realized pair of signals resulting in the posterior belief $\check{p}_0$, each agent's equilibrium payoff is $v_l^*(\check{p}_0)$ (when ρ^* is chosen so that $p_0(b,b) > p_l^*$ for all $\rho \in (1/2,\rho^*)$. Thus, the expected welfare under public information can be written as

$$\begin{aligned}E[W_\theta(s_1,s_2)] &= P(g,g)v_l^*(p_0(g,g)) + P(b,g)v_l^*(p_0)\\ &\quad + (P(g,b)+P(b,b))\Big[q_0(b)v_l^*(p_0) + (1-q_0(b))v_l^*(p_0(b,b))\Big]. \quad (28)\end{aligned}$$

Using the definition of τ^* in Lemma 1, we have

$$v_l^*(p) = v_l(p,\tau^*(p)) = py+(1-p)\lambda_1(y-\gamma c)+(1-p)e^{-(r+\gamma)\tau^*(p)}(\lambda_2-\lambda_1)(y-\gamma c)-rI.$$

Since $p_0 = p_f^*$, we have $\tau^*(p_0) = \tau^*(p_0(g,g)) = 0$. We have that $\tilde{W} >$

$E[W_\theta(s_1, s_2)]$ if

$$\eta v_l(p_0(b), 0) + (1-\eta) v_l(p_0(b), \tau^*(\tilde{p}(\eta, b)) > q_0(b) v_l(p_0, 0) + (1-q_0(b)) v_l^*(p_0(b, b))). \tag{29}$$

We define $\psi(p) := \frac{p}{1-p} = e^{\phi(p)}$ and $\alpha := \frac{r+\gamma}{\gamma} > 1$. Then, the left-hand side of Inequality (29) can be written as

$$\eta v_l(p_0(b), 0) + (1-\eta) v_l(p_0(b), \tau^*(\tilde{p}(\eta, b)) = p_0(b) y + (1-p_0(b)) \lambda_1 (y - \gamma c)$$
$$+ (1 - p_0(b)) \left[\eta + (1-\eta) \psi(\tilde{p}(\eta, b))^\alpha \psi(p_f^*)^{-\alpha} \right] (\lambda_2 - \lambda_1)(y - \gamma c) - rI \tag{30}$$

From Bayes' rule and the definition of ψ, we have

$$\psi(p_0(b)) = \frac{p_0}{1-p_0} \frac{1-\rho}{\rho} = \psi(p_0)/\psi(\rho), \;\; \psi(\tilde{p}(\eta, b)) = \psi(p_0(b)) \left(\frac{\rho + (1-\rho)\eta}{1 - \rho + \rho\eta} \right).$$

If we now use the previous equalities to factor out $\psi(p_0(b))^\alpha \psi(p_f^*)^{-\alpha}$ from the square brackets in (30), we obtain

$$\eta v_l(p_0(b), 0) + (1-\eta) v_l(p_0(b), \tau^*(\tilde{p}(\eta, b)) = p_0(b) y + (1-p_0(b)) \lambda_1 (y - \gamma c)$$
$$+ (1-p_0(b)) \psi(p_0(b))^\alpha \psi(p_f^*)^{-\alpha} \left[\eta \psi(\rho)^\alpha + (1-\eta) \left(\frac{\rho + (1-\rho)\eta}{1-\rho+\rho\eta} \right)^\alpha \right] (\lambda_2 - \lambda_1)(y - \gamma c) - rI. \tag{31}$$

The right-hand side of Inequality (29) is given by

$$q_0(b) v_l(p_0, 0) + (1 - q_0(b)) v_l^*(p_0(b, b)) = p_0(b) y + (1 - p_0(b)) \lambda_1 (y - \gamma c)$$
$$+ \left[q_0(b)(1-p_0) + (1-q_0(b))(1-p_0(b,b)) \psi(p_0(b,b))^\alpha \psi(p_f^*)^{-\alpha} \right] (\lambda_2 - \lambda_1)(y - \gamma c) - rI. \tag{32}$$

From Bayes' rule it follows that

$$\frac{q_0(b)}{1-p_0(b)} = \frac{p_0(b)\rho + (1-p_0(b))(1-\rho)}{1-p_0(b)} = \left(\frac{p_0}{1-p_0}\frac{1-\rho}{\rho}\right)\rho + (1-\rho) = \frac{1-\rho}{1-p_0},$$
$$\frac{1-q_0(b)}{1-p_0(b)} = \frac{p_0(b)(1-\rho) + (1-p_0(b))\rho}{1-p_0(b)} = \left(\frac{p_0}{1-p_0}\frac{1-\rho}{\rho}\right)(1-\rho) + \rho = \frac{\rho}{1-p_0(b,b)}.$$

Using these equalities together with the identity

$$\psi(p_0(b,b)) = \frac{p_0(b)}{1-p_0(b)}\frac{1-\rho}{\rho} = \psi(p_0(b))\psi(1-\rho)$$

to factor out $(1-p_0(b))\psi(p_0(b))^\alpha\psi(p_f^*)^{-\alpha}$ from the square brackets in (32), we obtain

$$q_0(b)v_l(p_0,0) + (1-q_0(b))v_l^*(p_0(b,b)) = p_0(b)y + (1-p_0(b))\lambda_1(y-\gamma c)$$
$$+(1-p_0(b))\psi(p_0(b))^\alpha\psi(p_f^*)^{-\alpha}\Bigg[(1-\rho)\psi(\rho)^\alpha+\rho\psi(\rho)^{-\alpha}\Bigg](\lambda_2-\lambda_1)(y-\gamma c)-rI.$$

Define the functions,

$$h(\eta,\rho) := \eta\psi(\rho)^\alpha+(1-\eta)\left(\frac{\rho+(1-\rho)\eta}{1-\rho+\rho\eta}\right)^\alpha, \;\; g(\rho) := (1-\rho)\psi(\rho)^\alpha+\rho\psi(\rho)^{-\alpha}.$$

Condition (29) is thus equivalent to $\inf_\eta h(\eta,\rho) > g(\rho)$. One calculates that the partial derivative of h at $\rho = 1/2$ is $\lim_{\rho\to 1/2}\partial_\rho h(\eta,\rho) = 4\alpha\left(2\eta^2-\eta+1\right)/(\eta+1)$. The function $\lim_{\rho\to 1/2}\partial_\rho h(\eta,\rho)$ has its minimum in η at $\sqrt{2}-1$ and is thus larger than $4\left(4\sqrt{2}-5\right) > 0$. On the other hand, $g'(1/2) = 0$. Thus, there exists a $\rho^* > 1/2$ such that for all $\rho \in (1/2,\rho^*)$, we have $\tilde{W} > E[W_\theta(s_1,s_2)]$.

□

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